STATISTICAL CONVERGENCE AND INFINITE MATRICES

SUMMARY

THESIS

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SUMMARY

The thesis consists of eight chapters. Chapter I is introductory and contains notations, background and a rèsumè of hitherto known results which have direct relation with our investigations. The central theme of the thesis is the concept of statistical convergence for single as well as double sequences. The idea of statistical convergence is a recent development in the theory of summability and matrix transformations.

Let K be a subset of the natural numbers $\mathbb{N}$ and let $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is given by

$$\delta(K) = \lim_n \frac{1}{n} \mid K_n \mid,$$

if the limit exists, where the vertical bars denote the cardinality of the enclosed set.

Statistical convergence depends on the notion of density of sets of natural numbers.

A real or a complex sequence $x = (x_k)$ is called statistically convergent to the number ℓ if for every $\varepsilon > 0$ the set

$$K(\varepsilon) = \{k \in K : \mid x_k - \ell \mid \geq \varepsilon\}$$

has natural density zero, i.e.

$$\lim_n \frac{1}{n} \mid \{k \leq n : \mid x_k - \ell \mid \geq \varepsilon\} \mid = 0.$$

In this case we write $st - \lim x = \ell$.

By the symbol st we denote the set of all statistically convergent sequences and by st_o the set of all statistically null sequences.

In Chapter II, we define and characterize statistically strongly regular and statistically almost regular matrices.

Let f denote the set of all almost convergent sequences. Then

$$f := \{x \in \ell_\infty : \lim_p \frac{1}{p+1} \sum_{j=0}^{p} x_{n+j} = L, \text{ uniformly in } n\},$$

A matrix A is said to be almost regular, i.e. $A \in (c, f)_{reg}$ if $Ax \in f$ for $x \in c$ with $f - \lim Ax = \lim x$. A is said to be strongly regular, i.e. $A \in (f, c)_{reg}$ if $Ax \in c$ for $x \in f$ with $\lim Ax = f - \lim x$, where f

We define the following:

An infinite matrix $A = (a_{nk})$ is said to be statistically almost regular if $Ax \in f$ for all $x \in st \cap \ell_\infty$ with $st - \lim x = f - \lim Ax$, where ℓ_∞ denotes the space of bounded sequences. We denote the class of such matrices by $(st \cap \ell_\infty, f)_{reg}$.

An infinite matrix A is said to be statistically strongly regular if $Ax \in st \cap \ell_\infty$ for all $x \in f$ with $f - \lim x = st - \lim Ax$. We denote the class of such matrices by $(f, st \cap \ell_\infty)_{reg}$.

Note that by Ax we mean the A-transform of the sequence $x = (x_k)$.

We prove the following theorems.

Theorem 2.1. $A \in (st \cap \ell_\infty, f)_{reg}$ if and only if A is almost regular and

(2.1.1) $\lim_p \sum_{k \in E} \mid t(n,k,p) \mid = 0$ uniformly in n for every $E \subseteq \mathbb{N}$

such that $\delta(E) = 0$; where

$$t(n,k,p) = \frac{1}{p+1} \sum_{i=0}^{p} a_{n+i,k}.$$

Theorem 2.2. $A \in (f, st \cap \ell_\infty)_{reg}$ if and only if

(2.2.1) $A \in (c, st \cap \ell_\infty)_{reg}$;

(2.2.2) there exists an index set $N = \{n_i\}$ such that $\delta(N) = 1$ and

$$\lim_i \sum_k \mid a_{n_i k} - a_{n_i,k+1} \mid = 0;$$

where c is the space of convergent sequences.

We use the above results to establish some core theorem in Chapter IV.

The Knopp core (or K-core) of a real bounded sequence x is defined to be the closed interval $[\ell(x), L(x)]$, where $\ell(x) = \liminf x$; $L(x) = \limsup x$.

The famous Knopp core theorem states that $L(Ax) \leq L(x)$ or $(K-\text{core}\{Ax\} \subseteq K-\text{core}\{x\})$ for every real bounded sequence x, if and only if A is regular and $\lim_n \sum_k \mid a_{nk} \mid = 1$.

If x is a statistically bounded sequence (i.e. there is a number B such that $\delta\{k : \mid x_k \mid > B\} = 0$), then the statistical core of x is defined to be the closed interval $[st-\liminf x,\ st-\limsup x]$.

A finite collection $\{K_1, K_2, \cdots, K_\ell\}$ of pairwise disjoint subset of $\mathbb{N}$ is called a statistical partition of $\mathbb{N}$ if (i) $\delta(\cup_{j=1}^{\ell} K_j) = 1$, (ii) $\delta^*(K_j) > 0$, for $j = 1, 2, \cdots, \ell$; where δ^* denotes the upper density which is defined by

$$\delta^*(K) = \limsup_n \frac{1}{n} \mid \{k \leq n : k \in K\} \mid .$$

$\{K_1, K_2, \cdots, K_\ell\}$ is called a superior partition of $\mathbb{N}$ if (1) $\mathbb{N} \backslash \cup_{j=1}^{\ell} K_j$ is finite, (2) K_j is infinite for each $j \leq \ell$.

In Chapter III, we prove the following theorems.

Theorem 3.1. If $\|A\| < \infty$, then for every $x \in \ell_\infty$

$$st-\text{core}\{Ax\} \subseteq K-\text{core}\{x\}$$

if and only if

(3.1.1) $st-\lim_n \sum_{k\in E} a_{nk} = 1$, whenever $\mathbb{N} \backslash E$ is finite, for $E \subseteq \mathbb{N}$; and

(3.1.2) $A \in (c, st \cap \ell_\infty)_{reg}$.

Theorem 3.2. If $\|A\| < \infty$, then for every $x \in \ell_\infty$

$$st - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}$$

if and only if

(3.2.1) $st - \lim_n \sum_{k \in E} a_{nk} = 1$, for every $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$; and

(3.2.2) $A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$.

Proposition 3.3. If $\|A\| < \infty$ and $A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$, then $st - \limsup_n \sum_{i=1}^{\ell} \mid \sum_{j \in K_i} a_{nj} \mid \leq 1$, whenever $\{K_1, K_2, \cdots, K_\ell\}$ is a st-partition of $\mathbb{N}$, but converse need not be true.

Proposition 3.4. If $\|A\| < \infty$ and $A \in (c, st \cap \ell_\infty)_{reg}$, then $st - \limsup_n \sum_{i=1}^{\ell} \mid \sum_{j \in K_i} a_{nj} \mid \leq 1$, whenever $\{K_1, K_2, \cdots, K_\ell\}$ is a sup-partition of $\mathbb{N}$, but converse need not be true.

These propositions show that our condition implies the condition of Li and Fridy.

The following theorems is more general than our Theorems 3.1 and 3.2.

Theorem 3.5. Let $T = (t_{jk})$ be a normal matrix (i.e. triangular with non-zero diagonal entries) and $A = (a_{nj})$ be any matrix. In order that whenever Tx is bounded Ax should exist and be bounded and satisfy

$$st - \text{core}\{Ax\} \subseteq K - \text{core}\{Tx\}$$

it is necessary and sufficient that

(3.5.1) $(c_{nk}) = C = AT^{-1}$ exists;

(3.5.2) $C \in (c, st \cap \ell_\infty)_{reg}$;

(3.5.3) $st - \lim_n \sum_{k \in E} c_{nk} = 1$, whenever $\mathbb{N} \backslash E$ is finite;

(3.5.4) for any fixed n,

$$\sum_{k=0}^{m} \mid \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} \mid \to 0 \text{ as } m \to \infty.$$

Theorem 3.6. Let A and T be same as in Theorem 3.5. Then

$$st-\text{core}\{Ax\} \subseteq st-\text{core}\{Tx\}$$

if and only if (3.5.1) and (3.5.4) hold and

(3.6.1) $C \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$;

(3.6.2) $st-\lim_n \sum_{k\in E} c_{nk} = 1$, for every $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$.

In Chapter IV, we establish the inclusions between the Banach core and the statistical core of a bounded sequence. The Banach core (or B-core) of a real bounded sequence x is defined to be the closed interval $[-q(-x), q(x)]$, where

$$q(x) = \limsup_p \sup_n t_{pn}(x)$$

is a sublinear functional on ℓ_∞.

We prove the following results.

Theorem 4.1. If $\|A\| < \infty$, then for every $x \in \ell_\infty$

$$B-\text{core}\{Ax\} \subseteq st-\text{core}\{x\}$$

if and only if

(4.1.1) A is almost regular, and

$$\lim_p \sum_{k\in E} \mid t(n,k,p) \mid = 0 \text{ uniformly in } n, \text{ whenever } \delta(E) = 0 \text{ for } E \subseteq \mathbb{N};$$

(4.1.2) $\limsup_p \sup_n \sum_k \mid t(n,k,p) \mid = 1$.

Theorem 4.2. Let $T = (t_{jk})$ be a normal matrix and $A = (a_{nj})$ be any matrix. In order that whenever Tx is bounded Ax should exist and be bounded and satisfy

$$B-\text{core}\{Ax\} \subseteq st-\text{core}\{Tx\}$$

it is necessary and sufficient that

(4.2.1) $(c_{nk}) = C = AT^{-1}$ exists;

(4.2.2) C is almost regular, and

$$\lim_p \sum_{k\in E} \mid b(n,k,p) \mid = 0 \text{ uniformly in } n, \text{ whenever } \delta(E) = 0 \text{ for } E \subseteq \mathbb{N};$$

(4.2.3) $\limsup_p \sup_n \sum_k \mid b(n,k,p) \mid = 1$, where

$$b(n,k,p) = \frac{1}{p+1}\sum_{i=0}^{p} c_{n+i,k} \ ;$$

(4.2.4) for any fixed n,

$$\sum_{k=0}^{m} \mid \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} \mid \to 0 \text{ as } m \to \infty \ .$$

Theorem 4.3. Let A and T be same as in Theorem 4.2. Then

$$B - \text{core}\{Ax\} \subseteq K - \text{core}\{Tx\}$$

if and only if (4.2.1) and (4.2.4) hold, and

(4.3.1) C is almost regular;

(4.3.2) $\limsup_p \sup_n \sum_k \mid \frac{1}{p+1}\sum_{i=0}^{p} c_{n+i,k} \mid = 1$.

Theorem 4.4. Let A and T be same as in Theorem 4.2. Then

$$B - \text{core}\{Ax\} \subseteq B - \text{core}\{Tx\}$$

if and only if (4.2.1), (4.2.4) and (4.3.2) hold, and

(4.4.1) C is F-regular, i.e. $C \in (f,f)_{reg}$.

Theorem 4.5. If $\|A\| < \infty$, then for every $x \in \ell_\infty$

$$st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\}$$

if and only if

(4.5.1) $A \in (f, st \cap \ell_\infty)_{reg}$;

(4.5.2) $st - \lim_n \sum_{k\in E} a_{nk} = 1$, whenever $\mathbb{N}\backslash E$ is finite for $E \subseteq \mathbb{N}$.

Theorem 4.6. Let A and T be same as in Theorem 4.2. Then

$$st - \text{core}\{Ax\} \subseteq B - \text{core}\{Tx\}$$

if and only if

(4.6.1) $(c_{nk}) = C = AT^{-1}$ exists;

(4.6.2) $C \in (f, st \cap \ell_\infty)_{reg}$;

(4.6.3) $st - \lim_n \sum_{k \in E} c_{nk} = 1$ whenever $\mathbb{N} \backslash E$ is finite;

(4.6.4) for any fixed n,

$$\sum_{k=0}^{m} \mid \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} \mid \to 0 \text{ as } m \to \infty.$$

In Chapter V, we generalize the concepts of statistical convergence, statistical boundedness, statistical cluster point, statistical limit point, statistical limit superior and limit inferior for a sequence of infinite matrices $\mathcal{B} = (B_i)$, where $B_i = (b_{nk}(i))$.

A sequence $x \in \ell_\infty$ is said to be $F_{\mathcal{B}}$-convergent (or $\mathcal{B}$-summable) to the value $\mathcal{B} - \lim x$ (denotes the generalized limit) if

$$\lim_n (B_i x)_n = \lim_n \sum_k b_{nk}(i) = \mathcal{B} - \lim x, \text{ uniformly in } i \geq 0.$$

By an index set we mean a subset $\{k_i\}$ of $\mathbb{N}$ with $k_i \leq k_{i+1}$.

An index set K is said to have $\mathcal{B}$-density $\delta_{\mathcal{B}}(K)$ equal to d, if

$$\lim_n \sum_{k \in K} b_{nk}(i) = d, \text{ uniformly in } i.$$

Let $\Re^+$ denote the set of all regular methods $\mathcal{B}$ with $b_{nk}(i) \geq 0$ for all n, k and i.

Let $\mathcal{B} \in \Re^+$. A sequence $x = (x_k)$ is called $\mathcal{B}$-statistically convergent to the number ℓ, if for every $\varepsilon > 0$

$$\delta_{\mathcal{B}} \mid \{k : \mid x_k - \ell \mid \geq \varepsilon\} \mid = 0$$

and we write $st_{\mathcal{B}} - \lim x = \ell$. We denote by $st(\mathcal{B})$ the space of all $\mathcal{B}$-statistically convergent sequences.

By $\delta_{\mathcal{B}}(K) \neq 0$ we mean that either $\delta_{\mathcal{B}}(K) > 0$ or K fails to have $\mathcal{B}$-density.

Let $\mathcal{B} \in \Re^{+}$. The number γ is said to be $\mathcal{B}$-statistical cluster point of a sequence x if for every $\varepsilon > 0$ the set $\{k : \mid x_k - \gamma \mid < \varepsilon\}$ does not have $\mathcal{B}$-density zero.

Let $\mathcal{B} \in \Re^{+}$. The number λ is said to be $\mathcal{B}$-statistical limit point of a sequence x if there is a subsequence of x which converges to λ such that whose indices do not have $\mathcal{B}$-density zero.

We denote by $\Gamma_x(\mathcal{B})$ the set of $\mathcal{B}$-statistical cluster points and by $\Lambda_x(\mathcal{B})$ the set of $\mathcal{B}$-statistical limit points of x.

Let us write

$$G_x = \{g \in \mathbb{R} : \delta_{\mathcal{B}}\{k : x_k > g\} \neq 0\},$$

and

$$F_x = \{f \in \mathbb{R} : \delta_{\mathcal{B}}\{k : x_k < f\} \neq 0\},$$

for a number sequence $x = (x_k)$. Then we define the $\mathcal{B}$-statistical limit superior and $\mathcal{B}$-statistical limit inferior of x as follows:

$$st_{\mathcal{B}} - \limsup x = \begin{cases} \sup G_x & , \text{ if } G_x \neq \emptyset, \\ -\infty & , \text{ if } G_x = \emptyset, \end{cases}$$

and

$$st_{\mathcal{B}} - \liminf x = \begin{cases} \inf F_x & , \text{ if } F_x \neq \emptyset, \\ +\infty & , \text{ if } F_x = \emptyset. \end{cases}$$

The number sequence x is said to be $\mathcal{B}-$ statistically bounded if there is a number M such that

$$\delta_{\mathcal{B}}\{k : \mid x_k \mid > M\} = 0.$$

We prove the following results.

Theorem 5.1. (a) If $s_1 = st_{\mathcal{B}} - \limsup x$ is finite, then for every positive number ε

(5.1.1) $\delta_{\mathcal{B}}\{k : x_k > s_1 - \varepsilon\} \neq 0$ and $\delta_{\mathcal{B}}\{k : x_k > s_1 + \varepsilon\} = 0$.

Conversely, if (5.1.1) holds for every $\varepsilon > 0$ then $s_1 = st_{\mathcal{B}} - \limsup x$.

(b) If $s_2 = st_{\mathcal{B}} - \liminf x$ is finite, then for every positive number ε

(5.1.2) $\delta_{\mathcal{B}}\{k : x_k < s_2 + \varepsilon\} \neq 0$ and $\delta_{\mathcal{B}}\{k : x_k < s_2 - \varepsilon\} = 0$.

Conversely, if (5.1.2) holds for every $\varepsilon > 0$ then $s_2 = st_{\mathcal{B}} - \liminf x$.

Theorem 5.2. For any real number sequence x

$$st_{\mathcal{B}} - \liminf x \leq st_{\mathcal{B}} - \limsup x.$$

Theorem 5.3. For any number sequence x,

$$\mathcal{B}\text{-statistical boundedness} \implies \mathcal{B}\text{-statistical convergence}$$

if and only if

$$st_{\mathcal{B}} - \liminf x = st_{\mathcal{B}} - \limsup x.$$

Theorem 5.4. If the number sequence x is bounded above and $\mathcal{B}$-summable to the number $\ell = st_{\mathcal{B}} - \limsup x$, then x is $\mathcal{B}$-statistically convergent to ℓ.

Theorem 5.5. If the number sequence x is bounded below and $\mathcal{B}$-summable to the number $\ell = st_{\mathcal{B}} - \liminf x$, then x is $\mathcal{B}$-statistically convergent to ℓ.

In Chapter VI, we establish a core theorem involving the concept of almost convergence for double sequences.

A double sequence $x = (x_{jk})_{j,k=0}^{\infty}$ is said to be convergent in the Pringsheim's sense or P-convergent if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\, x_{jk} - \ell \,| < \varepsilon$ whenever $j, k > N$ and we denote by $P - \lim x = \ell$.

We denote the space of P-convergent sequences by c_2.

A double sequence x is <u>bounded</u> if there exists a positive number M such that $| x_{jk} | < M$ for all j and k, i.e. if

$$\|x\|_{(\infty,2)} = \sup_{j,k} | x_{jk} | < \infty .$$

We denote the set of all bounded double sequences by ℓ^2_∞.

A double sequence $x = (x_{jk})_{j,k=0}^{\infty}$ of real numbers is said to be <u>almost convergent</u> to a limit L if

$$\lim_{p,q\to\infty} \sup_{m,n\geq 0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - L \right| = 0 .$$

We say that a four dimensional matrix $A = (a_{jk}^{mn})$ is <u>strongly regular</u> if every almost convergent double sequence $x = (x_{jk})$ is A-summable to the same limit, and the A-means

$$y_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} \, x_{jk} ,$$

are also bounded.

We write

$$L^*(x) = \limsup_{p,q\to\infty} \sup_{m,n\geq 0} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} .$$

Then we define the MR-core of a real-valued bounded double sequence x to be the closed interval $[-L^*(-x), L^*(x)]$.

Also the Pringsheim core or P-core of a real-valued bounded double sequence is the closed interval $[P - \liminf x, P - \limsup x]$.

Note that

$$MR\text{-core}\{x\} \subseteq P\text{-core}\{x\}$$

The following result was given by Patterson.

Theorem 6.1. If A is a four dimensional matrix, then for all real-valued double sequences x ,

$$P - \limsup Ax \leq P - \limsup x$$

if and only if

(6.1.1) A is bounded regular;

(6.1.2) $P - \lim_{mn} \sum_{j,k=0,0}^{\infty,\infty} \mid a_{jk}^{mn} \mid = 1$.

We prove the following theorem which is a generalization of the above theorem.

Theorem 6.2. Let A be a four dimensional matrix. For every bounded double sequence x,

$$P\text{-core}\{Ax\} \subseteq MR\text{-core}\{x\}$$

if and only if

(6.2.1) $A = (a_{jk}^{mn})$ is strongly regular; and

(6.1.2) holds.

In this chapter we also construct various examples, e.g. almost convergent double sequences, strongly regular four dimensional matrix, bounded-regular matrix which is not strongly regular.

In Chapter VII. we define and study statistical convergence for double sequences.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K(n,m)$ be the numbers (i,j) in K such that $i \leq n$ and $j \leq m$. Then the double natural density of K is defined by

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K).$$

A real double sequence $x = (x_{jk})$ is said to be statistically convergent to the number ℓ if for each $\varepsilon > 0$, the set

$$\{(j,k), j \leq n \ \text{ and } \ k \leq m : \mid x_{jk} - \ell \mid \geq \ \varepsilon\}$$

has double natural density zero. In this case we write $st_2 - \lim_{n,m} x_{nm} = \ell$ and we denote the set of all statistically convergent double sequences by st_2.

We prove the following results.

Theorem 7.1. A real double sequence $x = (x_{jk})$ is statistically convergent to a number ℓ if and only if there exists a subset $K = \{(i,s)\} \subseteq \mathbb{N} \times \mathbb{N}$, $i, s = 1, 2, \cdots$ such that $\delta(K) = 1$ and

$$\lim_{i,s} x_{j_i, k_s} = \ell.$$

Theorem 7.2. The set $st_2 \cap \ell^2_\infty$ is a closed linear subspace of the normed linear space ℓ^2_∞.

Theorem 7.3. The set $st_2 \cap \ell^2_\infty$ is nowhere dense in ℓ^2_∞.

We also define the following.

A real double sequence $x = (x_{jk})$ is said to be statistically Cauchy if for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that the set

$$\{(j,k), j \leq n, k \leq m : | x_{jk} - x_{NM} | \geq \varepsilon\}$$

has double natural density zero.

Theorem 7.4. A real double sequence $x = (x_{jk})$ is statistically convergent if and only if x is statistically Cauchy.

Theorem 7.5. The following statements are equivalent:

(7.5.1) x is statistically convergent

(7.5.2) x is statistically Cauchy

(7.5.3) there exists a set $K_{nm} = \{(j_1, k_1), \cdots, (j_n, k_m)\}$ such that $\delta_2(K_{nm}) = 1$ and $\lim_{n,m} x_{j_n, k_m} = 1$.

Corollary 7.6. If x is statistically convergent to ℓ then there exists a subsequence y of x such that

$$\lim y = \ell \text{ and } \delta_2\{(j,k) : x_{jk} = y_{jk}\} = 1.$$

We also establish the relation between statistical convergence and strongly Cesàro summable sequences.

We say that a double sequence $x = (x_{jk})$ is C_{11}-summable or Cesàro summable to a finite limit ℓ if the sequence (σ_{mn}^{11}) is convergent to ℓ in Pringsheim's sense, i.e.

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = \ell.$$

Similarly C_{10} and C_{01} summable sequences are defined, where

$$\sigma_{mn}^{11} = \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk},$$

$$\sigma_{mn}^{10} = \frac{1}{m} \sum_{j=1}^{m} x_{jn},$$

and

$$\sigma_{mn}^{01} = \frac{1}{n} \sum_{k=1}^{n} x_{mk}.$$

Let $x = (x_{jk})$ be a double sequence and p be a positive real number. Then the double sequence x is said to be strongly p-Cesàro summable to ℓ if

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} \mid x_{jk} - \ell \mid^{p} = 0.$$

We denote the space of all strongly p-Cesàro summable double sequences by w_p^2.

Theorem 7.7. Let $x = (x_{jk})$ be a double sequence and p be a positive real number. Then

(a) if x is strongly p-Cesàro summable to ℓ, then it is also statistically convergent to ℓ,

(b) $w_p^2 \cap \ell_\infty^2 = st_2 \cap \ell_\infty^2$.

In Chapter VIII, we prove some Tauberian theorems for statistically convergent double sequences.

We denote the backward differences of x_{jk} as follows:

$$\nabla_{11}\ x_{jk} = x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}$$

$$\nabla_{10}\ x_{jk} = x_{jk} - x_{j-1,k}$$

$$\nabla_{01}\ x_{jk} = x_{jk} - x_{j,k-1}.$$

We can also easily compute the following differences:

D_1. $$x_{jk} - x_{j+p,k+q} = \sum_{i=j+1}^{j+p} \sum_{s=k+1}^{k+q} \nabla_{11}\ x_{is} - \sum_{i=j+1}^{j+p} \nabla_{10}\ x_{i,k+q} - \sum_{s=k+1}^{k+q} \nabla_{01}\ x_{j+p,s}$$
$$= -\sum_{i=j+1}^{j+p} \nabla_{10}\ x_{ik} - \sum_{s=k+1}^{k+q} \nabla_{01}\ x_{j+p,s}$$

D_2. $$x_{nj} - x_{nk} = \sum_{s=k+1}^{j} \nabla_{01}\ x_{ns}$$

D_3. $$x_{nm} - x_{km} = \sum_{i=k+1}^{n} \nabla_{10}\ x_{im}$$

D_4. $$x_{jk} - x_{nm} = \sum_{i=j+1}^{n} \sum_{s=k+1}^{m} \nabla_{11}\ x_{is} - \sum_{i=j+1}^{n} \nabla_{10}\ x_{im} - \sum_{s=k+1}^{m} \nabla_{01}\ x_{ns}$$

D_5. $$x_{jk} - x_{nm} - x_{nk} + x_{nm} = \sum_{i=j+1}^{n} \sum_{s=k+1}^{m} \nabla_{11}\ x_{is}.$$

Theorem 8.1. Let $x = (x_{jk})$ be a double sequence with $st_2 - \lim x = \ell$ and there exist a constant n_1 such that

(8.1.1) $\nabla_{11}\ x_{jk} = O(\frac{1}{jk})$;

(8.1.2) $\nabla_{10}\ x_{jk} = O(\frac{1}{j})$;

(8.1.3) $\nabla_{01}\ x_{jk} = O(\frac{1}{k})$,

whenever $j, k > n_1$. Then $\lim_{j,k} x_{jk} = \ell$.

Lemma 8.2. Let $x = (x_{jk})$ be a double sequence. If there exists a constant n_1 such that the following conditions hold

(8.2.1) $\nabla_{11}\ x_{jk} = O(\frac{1}{jk})$;

(8.2.2) $\nabla_{10}\ x_{jk} = O(\frac{1}{j})$;

(8.2.3) $\nabla_{01}\ x_{jk} = O(\frac{1}{k})$,

whenever $j, k > n_1$. Then $(\nabla_{11}\ C_{11}\ x)_{nm} = O(\frac{1}{nm})$.

Corollary 8.3. Let $x = (x_{jk})$ be a double sequence and there exist a constant n_1 such that

(a) if $\nabla_{10}\ x_{jk} = O(\frac{1}{j})$, then $(\nabla_{10}\ C_{10}\ x)_n = O(\frac{1}{n})$

(b) if $\nabla_{01}\ x_{jk} = O(\frac{1}{k})$, then $(\nabla_{01}\ C_{01}\ x)_m = O(\frac{1}{m})$,

whenever $j, k > n_1$.

Theorem 8.4. For a double sequence $x = (x_{jk})$ if $st_2 - \lim C_{11}\ x = \ell$ and there exists a constant n_1 such that

(8.4.1) $\nabla_{11}\ x_{jk} = O(\frac{1}{jk})$;

(8.4.2) $\nabla_{10}\ x_{jk} = O(\frac{1}{j})$;

(8.4.3) $\nabla_{01}\ x_{jk} = O(\frac{1}{k})$,

whenever $j, k > n_1$. Then $\lim_{j,k} x_{jk} = \ell$.

Corollary 8.5. Let $x = (x_{jk})$ be a double sequence and there exist a constant n_1 such that

(a) if $st_2 - \lim C_{10}\ x = \ell$ and $\nabla_{10}\ x_{jk} = O(\frac{1}{j})$ then $\lim_{j,k} x_{jk} = \ell$

(b) if $st_2 - \lim C_{01}\ x = \ell$ and $\nabla_{01}\ x_{jk} = O(\frac{1}{k})$ then $\lim_{j,k} x_{jk} = \ell$,

whenever $j, k > n_1$.

Theorem 8.6. Let $x = (x_{jk})$ be a double sequence with $st_2 - \lim x = \ell$ and there exist constants n_1 and M such that

(8.6.1) $jk\nabla_{11}\ x_{j+1,k+1} \geq -M$;

(8.6.2) j $\nabla_{10}\ x_{j+1,k} \geq -M$;

(8.6.3) k $\nabla_{01}\ x_{j,k+1} \geq -M$,

whenever $j, k > n_1$. Then $\lim_{j,k} x_{jk} = \ell$.

Lemma 8.7. Let $x = (x_{jk})$ be a double sequence and there exist constants n_1 and M such that

(8.7.1) $jk\nabla_{11}\ x_{j+1,k+1} \geq -M$;

(8.7.2) j $\nabla_{10}\ x_{j+1,k} \geq -M$;

(8.7.3) k $\nabla_{01}\ x_{j,k+1} \geq -M$,

whenever $j, k > n_1$. Then $(\nabla_{11}\ C_{11}\ x)_{nm} \geq -M$.

Corollary 8.8. Let $x = (x_{jk})$ be a double sequence and there exist constants n_1 and M such that

(a) if $j\nabla_{10}\ x_{j+1,k} \geq -M$, then $(\nabla_{10}\ C_{10}\ x)_n \geq -M$

(b) if $k\nabla_{01}\ x_{j,k+1} \geq -M$, then $(\nabla_{01}\ C_{01}\ x)_m \geq -M$,

whenever $j, k > n_1$.

Theorem 8.9. Let $x = (x_{jk})$ be a double sequence with $st_2 - \lim C_{11}\ x = \ell$ and there exist constants n_1 and M such that

(8.9.1) $jk\nabla_{11}\ x_{j+1,k+1} \geq -M$;

(8.9.2) j $\nabla_{10}\ x_{j+1,k} \geq -M$;

(8.9.3) k $\nabla_{01}\ x_{j,k+1} \geq -M$,

whenever $j, k > n_1$. Then $\lim_{j,k} x_{jk} = \ell$.

Corollary 8.10. Let $x = (x_{jk})$ be a double sequence and there exist constants n_1 and M such that

(a) if $st_2 - \lim C_{10}\, x = \ell$ and $j\nabla_{10}\, x_{j+1,k} \geq -M$ then $\lim_{j,k} x_{jk} = \ell$

(b) if $st_2 - \lim C_{01}\, x = \ell$ and $k\nabla_{01}\, x_{j,k+1} \geq -M$ then $\lim_{j,k} x_{jk} = \ell$,

whenever $j, k > n_1$.

The main feature of the present work is that each new concept and almost every result is supported by some numerical examples.

STATISTICAL CONVERGENCE AND INFINITE MATRICES

THESIS

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IN

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Certificate

This is to certify that the contents of this thesis entitled ***"STATISTICAL CONVERGENCE AND INFINITE MATRICES"*** *is an original research work of* ***Mr. Osama Hasan Husein Edely*** *carried out under my supervision. He has fulfilled the prescribed conditions given in the ordinances and regulations of Aligarh Muslim University, Aligarh.*

I further certify that the work has not been submitted either partly or fully to any other University or Institution for the award of any other degree.

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Dedicated

to

My Lovely Parents,

Brothers

&

Sisters

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PREFACE

The present thesis entitled,"STATISTICAL CONVERGENCE AND INFINITE MATRICES", is an outcome of my research that I have been pursuing since 9.8.1999, under the esteemed supervision of Dr. Mursaleen, Reader, Department of Mathematics, Aligarh Muslim University, Aligarh.

The central theme of the thesis is the concept of statistical convergence for single as well as double sequences through which we have established results of various natures, e.g. limit point and cluster point, limit superior and limit inferior, matrix transformations, core theorems,Tauberian theorems etc.

The thesis consists of eight chapters. In Chapter I, we recall some elementary definitions, notations and background material. Chapter II concerns with the study to characterize some matrix classes involving the space of statistically convergent sequences and we use these matrix classes in the subsequent chapters to establish core theorems.

Chapter III and IV are devoted to establish core theorems, i.e. inclusions involving the Knopp core, Banach core and statistical core.

In Chapter V, we generalize the concept of statistical convergence and other related concepts through a sequence of infinite matrices $\mathcal{B} = (B_i)$. This chapter is also a bridge between two passages of the study of single and double sequences.

Chapter VI deals with the study of double sequences. We define here MR-core of a double sequence involving the idea of almost convergence for double sequences; and we find necessary and sufficient conditions to establish the inclusion between P-core (Pringsheim's core) and MR-core. There and onward we consider the four dimensional infinite matrices.

In Chapter VII, we introduce and discuss the concepts of statistically convergent and statistically Cauchy double sequences $x = (x_{jk})$ and establish the relation between them. We also find the relation between statistical convergence and strongly Cesàro summable sequences.

VIII th and the last chapter is devoted to establish various interesting Tauberian theorems for statistically convergent double sequences.

The main feature of the present work is that each new concept and almost every result is supported by some numerical examples.

Towards the end of the thesis, we have given a fairly exhaustive bibliography of the books and publications to which references have been made throughout the thesis.

Osama

September 6, 2001. (Osama Hasan Husein Edely)

CONTENTS

CHAPTER I

BACKGROUND AND INTRODUCTION

1.1. Historical Note

The concept of statistical convergence was first introduced by Fast [14] in 1951. In 1953, this concept arises as an example of "convergence in density" as introduced by Buck [3]. Schoenberg [51] in 1959, studied statistical convergence as a summability method and gave some properties of statistical convergence. Zygmund [55] established a relation between it and strong summability. The concept of convergence in density, where the "density" is generated by matrix and non-matrix summability methods, has been explored by Freedman and Sember ([15], [16], [17]). In the recent years since 1980 this idea has grown a little fast after the papers of Šalát [49] and Fridy [18]. Fridy [18] has shown that statistical convergence is a non-matrix method in the sense that it is not included by any regular matrix method.

1.2. Notations

Throughout the present work we shall use the following notations which are conventional (cf. Cook [9], Hardy [26], Maddox [36]).

$\mathbb{N}$:= The set of all natural numbers

$\mathbb{R}$:= The set of all real numbers

$\mathbb{C}$:= The set of all complex numbers

$\lim_k$: means $\lim_{k\to\infty}$

$\inf_k$: means $\inf_{k\geq 1}$, unless otherwise stated

$\sup_k$: means $\sup_{k\geq 1}$, unless otherwise stated

$\sum_k$: means summation over $k=1$ to $k=\infty$, unless otherwise stated

$x=(x_k)$ or $\{x_k\}$, the sequence whose k-th term is x_k

$e_k=(0,0,\cdots,0,1,0,0,\cdots)$, the sequence whose k-th component is 1 and others zeros, for all $k\in\mathbb{N}$

$e=(1,1,1,\cdots)$

$\omega := \{x=(x_k) : x_k\in\mathbb{R}$ or $\mathbb{C}\}$, the space of all sequences, real or complex

$\ell_\infty := \{x\in\omega : \sup_k \mid x_k\mid <\infty\}$, the space of all bounded sequences

$c := \{x\in\omega : \lim_k x_k=\ell$ for some $\ell\in\mathbb{C}\}$ the space of all convergent sequences

$c_o := \{x\in\omega : \lim_k x_k=0\}$ the space of all null sequences

ℓ_∞, c and c_o are Banach spaces with the norm

$$\|x\|_\infty=\sup_k \mid x_k\mid .$$

$C_1 := \{x\in\omega : \lim_n \frac{1}{n}\sum_{k=1}^n x_k=\ell$ for some $\ell\in\mathbb{C}\}$ the space of all Cesàro summable sequences

$w_p := \{x\in\omega : \lim_n \frac{1}{n}\sum_{k=1}^n \mid x_k-\ell\mid^p=0$ for some $\ell\in\mathbb{C}\}$ the space of all strongly Cesàro summable sequences, where $0<p<\infty$

Cesàro matrix or C_1-matrix : A matrix $A = (a_{nk})$ such that

$$a_{nk} = \begin{cases} \frac{1}{n} & , 1 \leq k \leq n, \\ 0 & , k > n; \end{cases}$$

is called a Cesàro matrix of order 1.

If g is a positive function of a variable which tends to a limit, then we write

$f = O(g)$: means $\mid f \mid < Mg$, where M is constant

$f = o(g)$: means $f/g \to 0$.

1.3. Statistical Convergence: Definitions and Examples

Let K be a subset of $\mathbb{N}$ and let $K_n = \{k \leq n : k \in K\}$.

Definition 1.3.1. The natural density of K (cf. Niven and Zuckerman [42]) is given by

$$\delta(K) = \lim_n \frac{1}{n} \mid K_n \mid,$$

if the limit exists, where the vertical bars denote the cardinality of the enclosed set.

For example, the set of even positive integers has natural density $\frac{1}{2}$ and the set of primes has natural density zero.

Notice that

$$\delta(K) = \lim_n \frac{1}{n} \mid K_n \mid = \lim_n (C_1 \chi_K)_n,$$

where χ_K denotes the characteristic sequence of K given by

$$(\chi_K)_i = \begin{cases} 0 & , \text{ if } i \notin K, \\ 1 & , \text{ if } i \in K. \end{cases}$$

Statistical convergence depends on the notion of density of sets of natural numbers.

The notion of statistical convergence was first introduced by Fast [14] and also independently by Buck [3] and Schoenberg [51] for real and complex sequences. Further this concept was studied by Šalát [49], Fridy [18], Connor [6], Kolk [29] and many others.

Definition 1.3.2. A real or a complex sequence $x = (x_k)$ is called statistically convergent to the number ℓ if for every $\varepsilon > 0$ the set

$$K(\varepsilon) = \{k \in K : | x_k - \ell | \geq \varepsilon\}$$

has natural density zero, i.e.

$$\lim_n \frac{1}{n} | \{k \leq n : | x_k - \ell | \geq \varepsilon\} | = 0.$$

In this case we write $st - \lim x = \ell$.

By the symbol st we denote the set of all statistically convergent sequences and by st_o the set of all statistically null sequences.

Note that every convergent sequence is statistically convergent to the same number, so that statistical convergence is a natural generalization of the usual convergence of sequences.

The sequence which converges statistically need not be convergent and also need not be bounded.

Example 1.3.3. Let $x = (x_k)$ be defined by

(i) $$x_k = \begin{cases} k \ , & \text{if } k \text{ is a square,} \\ 0 \ , & \text{otherwise.} \end{cases}$$

We see that x is statistically convergent to zero but x is neither convergent nor bounded.

$$(ii) \qquad x_k = \begin{cases} 1 & , \text{ if } k \text{ is even,} \\ 2 & , \text{ if } k \text{ is odd.} \end{cases}$$

Here x is not statistically convergent.

By an index set we mean a subset $\{k_i\}$ of $\mathbb{N}$ with $k_i \leq k_{i+1}$.

Let $K = \{k_i\}$ be an index set. The K-section of a sequence $x = (x_k)$ is defined to be the sequence $x^{[K]} = (y_k)$, where

$$y_k = \begin{cases} x_k & , k \in K, \\ 0 & , \text{otherwise.} \end{cases}$$

A sequence space X will be called section-closed if $x^{[K]} \in X$ for all $x \in X$ and for every index set K.

Fridy [18] introduced the statistical analogue of the Cauchy convergence criterion.

Definition 1.3.4. The number sequence $x = (x_k)$ is called statistically Cauchy sequence if for every $\varepsilon > 0$ there exists a number $N(= N(\varepsilon))$ such that

$$\lim_n \frac{1}{n} \mid \{k \leq n : \mid x_k - x_N \mid \geq \varepsilon\} \mid = 0.$$

Fridy [18] proved the equivalence relation between Definition 1.3.2 and Definition 1.3.4.

Analogue to the definition of a limit point, Fridy [19] defined the following:

Definition 1.3.5. A subsequence $(x_{k(j)})$ of x is said to be a thin subsequence if $K = \{k(j) : j \in \mathbb{N}\}$ has natural density zero. It is called a nonthin subsequence of x if K does not have natural density zero.

Note that $(x_{k(j)})$ is a nonthin subsequence of x if either $\delta(K) > 0$ or K fails to have natural density.

Definition 1.3.6. The number λ is a statistical limit point of the number sequence x provided that there is a nonthin subsequence of x that converges to λ.

Definition 1.3.7. The number γ is a statistical cluster point of the number sequence x provided that for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : | x_k - \gamma | < \varepsilon\}$ does not have natural density zero.

Fridy [19] noted that for any number sequence x,

$$\Lambda_x \subseteq \Gamma_x \subseteq L_x,$$

where Λ_x, Γ_x and L_x denote the set of all statistical limit points, the set of all statistical cluster points and the set of ordinary limit points of x, respectively.

In [22], Fridy and Orhan introduced the following:

For a real number sequence x, let

$$B_x := \{b \in \mathbb{R} : \delta\{k : x_k > b\} \neq 0\},$$

and

$$A_x := \{a \in \mathbb{R} : \delta\{k : x_k < a\} \neq 0\} .$$

Definition 1.3.8. If x is a real number sequence, then the statistical limit superior of x is given by

$$st - \limsup x := \begin{cases} \sup B_x & , \text{ if } B_x \neq \emptyset, \\ -\infty & , \text{ if } B_x = \emptyset. \end{cases}$$

Also. the statistical limit inferior of x is given by

$$st - \liminf x := \begin{cases} \inf A_x & , \text{ if } A_x \neq \emptyset, \\ +\infty & , \text{ if } A_x = \emptyset. \end{cases}$$

Definition 1.3.9. The real number sequence x is said to be statistically bounded if there is a number B such that

$$\delta\{k : | x_k | > B\} = 0.$$

Example 1.3.10. Let $x = (x_k)$ be given by

$$x_k = \begin{cases} k & , \text{ if } k \text{ is a prime,} \\ 1 & , \text{ if } k \text{ is an odd not prime,} \\ 0 & , \text{ if } k \text{ is an even not prime.} \end{cases}$$

It is easy to see that x is not statistically convergent but it is statistically bounded since, $\delta\{k : \mid x_k \mid > 1\} = 0$, although x is unbounded above. Thus

$$B_x = (-\infty, 1), \ A_x = (0, \infty),$$

also

$$\Lambda_x = \{0, 1\}, \ \Gamma_x = \{0, 1\} \text{ and } L_x = \{0, 1\}.$$

Li and Fridy [34] introduced the concepts of statistical partition and superior partition of $\mathbb{N}$.

Definition 1.3.11. A finite collection $\{K_1, K_2, \cdots, K_\ell\}$ of pairwise disjoint subsets of $\mathbb{N}$ is called a statistical partition of $\mathbb{N}$ if the following conditions hold:

(1) $\delta(\bigcup_{j=1}^{\ell} K_j) = 1$;

(2) $\delta^*(K_j) > 0$, for $j = 1, 2, \cdots, \ell$.

By $\delta^*(K)$, where $K \subseteq \mathbb{N}$, is called the upper asymptotic density (cf. Halberstem [24]), which is defined by

$$\delta^*(K) = \limsup_n \frac{1}{n} \mid \{k \leq n : k \in K\} \mid .$$

Definition 1.3.12. A finite collection $\{K_1, K_2, \cdots, K_\ell\}$ of pairwise disjoint subsets of $\mathbb{N}$ is called a superior partition of $\mathbb{N}$ provided that the following conditions hold:

(1) $\mathbb{N} \setminus \bigcup_{j=1}^{\ell} K_j$ is finite;

(2) K_j is finite for each $j \leq \ell$.

For convenience we shall abbreviate statistical partition and superior partition by *st*-partition and *sup*-partition respectively.

For example, a *st*-partition of $\mathbb{N}$ is $\{E, O\}$, the subsets of even and odd integers respectively.

1.4. Some Useful Results

We note here some important results which will be used throughout the thesis, other results will be introduced as they become necessary.

Proposition 1.4.1 (Fridy [19]). If x is a bounded number sequence, then x has a statistical cluster point.

Proposition 1.4.2 (Li and Fridy [34]). If x is a statistically bounded complex sequence, then

$$\Gamma_x \subseteq st - \text{core}\{x\},$$

where $st - \text{core}\{x\}$ means the closed interval $[st - \liminf x, st - \limsup x]$.

Theorem 1.4.3 (Li and Fridy [34]). If $x^1, \cdots, x^\ell$ are statistically bounded complex sequences such that

$$st - \limsup(\mid x^1 \mid + \cdots + \mid x^\ell \mid) = \lambda,$$

then each x^i has a statistical cluster point α_i such that

(1) $\mid \alpha_1 \mid + \cdots + \mid \alpha_\ell \mid = \lambda;$

(2) for every $\varepsilon > 0$, $\delta^*\{n \in \mathbb{N} : \mid x_n^1 - \alpha_1 \mid + \cdots + \mid x_n^\ell - \alpha_\ell \mid < \varepsilon\} > 0.$

Proposition 1.4.4 (Li and Fridy [34]). If x is a statistically bounded complex sequence, then $st - \lim x = \gamma$ if and only if γ is the only statistical cluster point of x.

Proposition 1.4.5 (Šalát [49], Fridy [18], Kolk [31]). A sequence $x = (x_k)$ is statistically convergent to ℓ if and only if there is an infinite index set $K = \{k_i\}$ such that $\delta(K) = 1$ and $\lim_i x_{k_i} = \ell$.

Theorem 1.4.6 (Decomposition Theorem, cf. Connor [6]). If $x \in \omega$ is strongly p-Cesàro summable or statistically convergent to ℓ, then there is a convergent sequence y and a statistically null sequence z such that y is convergent to ℓ, $x = y + z$ and

$$\lim_n n^{-1} \mid \{k \leq n : z_k \neq 0\} \mid = 0.$$

Moreover, if x is bounded then z is also bounded and

$$\|z\|_\infty \leq \|x\|_\infty + \mid \ell \mid .$$

1.5. Almost Convergence

A continuous linear functional ϕ on ℓ_∞ is said to be a Banach limit (cf. Banach [1]) if

(i) $\phi(x) \geq 0$, when $x_n \geq 0$ for all n;

(ii) $\phi(e) = 1$;

(iii) $\phi(\{x_{n+1}\}) = \phi(\{x_n\})$.

Note that every Banach limit ϕ extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$.

We say that a sequence x is almost convergent (cf. Lorentz [35]) to some number L if all of its Banach limits coincide with L. Let f denote the set of all almost convergent sequences. Then

$$f := \{x \in \ell_\infty : \lim_p t_{pn}(x) = L, \text{ uniformly in } n\},$$

where $L = f - \lim x$, which is called the almost limit or generalized limit of x; and

$$t_{pn}(x) = \frac{1}{p+1}\sum_{j=0}^{p} x_{n+j}.$$

Note that the functional

$$q(x) = \lim_{p}\sup\sup_{n}\frac{1}{p+1}\sum_{j=0}^{p} x_{n+j}$$

is a sublinear functional on ℓ_∞. If

$$q(x) = -q(-x) = L,$$

then x is almost convergent to L.

1.6. Relation Between Statistical Convergence and Almost Convergence

We note that if $x \in st \cap \ell_\infty$, then $x \in f$ but not conversely.

For example the sequence $x = (x_k)$ with

$$x_k = \begin{cases} 1 & , \text{ if } k \text{ is odd,} \\ 0 & , \text{ if } k \text{ is even;} \end{cases}$$

is almost convergent to $\frac{1}{2}$ but not statistically convergent.

In [6] Connor proved that $st \cap \ell_\infty = w_p \cap \ell_\infty$.

Since, if a sequence is almost convergent to ℓ then it must be Cesàro summable to ℓ , we have

$$w_p \cap \ell_\infty = st \cap \ell_\infty \subseteq f \subseteq C_1.$$

1.7. Introduction

The central theme of the present thesis is to study statistical convergence for single as well as for double sequences.

In the first half of the thesis we characterize some matrix classes involving the space st and use these classes to establish some core theorems. In the second half we consider double sequences and generalize some previous results on almost convergence and statistical convergence for double sequences, like core theorems and Tauberian theorems.

CHAPTER II

STATISTICALLY STRONGLY AND STATISTICALLY ALMOST REGULAR MATRICES

2.1. Introduction

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex entries a_{nk} . By $Ax = (A_n(x))$ we denote the A-transform of the sequence $x = (x_k)_{k=1}^{\infty}$, where

$$A_n(x) = \sum_k a_{nk}x_k, \quad n = 1, 2, \cdots$$

provided that the series on the right hand side converges for each n. For any two sequence spaces X and Y, we write (X, Y) for a class of matrices A such that $Ax \in Y$ for $x \in X$. If in addition $\lim Ax = \lim x$, then we denote such a class by $(X, Y; P)$ or $(X, Y)_{reg}$.

We recall the characterizations of some useful matrix classes (cf. Cook [9], Maddox [36], Stieglitz and Tietz [54]).

Lemma 2.1.1. $A \in (c, c)$, i.e. A is conservative if and only if

(2.1.1.1) $\|A\| = \sup_n \sum_k \mid a_{nk} \mid < \infty$;

(2.1.1.2) $\lim_n a_{nk} = \alpha_k$ for each k;

(2.1.1.3) $\lim_n \sum_k a_{nk} = \alpha$.

The matrix A is said to be regular, i.e. $A \in (c, c)_{reg}$ if $Ax \in c$ for $x \in c$ with $\lim Ax = \lim x$. The following are famous Silverman-Töplitz conditions for the regularity of A:

Theorem 2.1.2. $A \in (c, c)_{reg}$ if and only if

(2.1.2.1) $\|A\| < \infty$;

(2.1.2.2) $\lim_n \sum_k a_{nk} = 1$;

(2.1.2.3) $\lim_n a_{nk} = 0$ for each k.

A matrix A is called uniformly regular if it satisfies the conditions (2.1.1.1), (2.1.2.2), and

$$\lim_n \sup_k \mid a_{nk} \mid = 0.$$

Lemma 2.1.3. $A \in (\ell_\infty, \ell_\infty)$ if and only if condition (2.1.1.1) holds.

Lemma 2.1.4. $A \in (\ell_\infty, c)$, i.e A is Schur if and only if

(2.1.4.1) for every fixed n, $\sum_{k=1}^{\infty} \mid a_{nk} \mid < \infty$;

(2.1.4.2) for every fixed k, $a_{nk} \longrightarrow \alpha_k$ as $n \longrightarrow \infty$;

(2.1.4.3) $\sum_{k=1}^{\infty} \mid a_{nk} - \alpha_k \mid \longrightarrow 0$ as $n \longrightarrow \infty$.

A triangular matrix with non-zero diagonal entries is said to be normal.

Lemma 2.1.5 (Choudhary [4]). Let $T = (t_{nk})$ be a normal matrix and $T^{-1} = (t_{nk}^{-1})$. Let $A = (a_{nk})$ be any matrix. Consider a fixed n. In order that, whenever Tx is bounded, $A_n(x)$ should be defined for that particular n, it is necessary and sufficient that

(2.1.5.1) $c_{nk} = \sum_{j=k}^{\infty} a_{nj} t_{jk}^{-1}$ exists for all k;

(2.1.5.2) $\sum_{k=1}^{\infty} \mid c_{nk} \mid < \infty$,

and that the following condition

(2.1.5.3) $\sum_{k=1}^{m} \mid \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} \mid \longrightarrow 0$ as $m \longrightarrow \infty$, should hold for the n considered. If these conditions are satisfied then, for bounded Tx,

$$A_n(x) = \sum_{k=1}^{\infty} c_{nk} y_k,$$

where

$$y_k = T_k(x) = \sum_{i=1}^{\infty} t_{ki} x_i \ .$$

The matrix A is said to be almost regular, i.e. $A \in (c, f)_{reg}$ if $Ax \in f$ for $x \in c$ with $f - \lim Ax = \lim x$. The following are the necessary and sufficient conditions for A to be almost regular (cf. King [27]):

Theorem 2.1.6. $A \in (c, f)_{reg}$ if and only if

(2.1.6.1) $\|A\| < \infty$;

(2.1.6.2) $\lim_p \frac{1}{p+1} \sum_k \sum_{i=0}^{p} a_{n+i,k} = 1$ uniformly in n;

(2.1.6.3) $\lim_p \frac{1}{p+1} \sum_{i=0}^{p} a_{n+i,k} = 0$ uniformly in n, for each k .

The matrix A is said to be strongly regular, i.e. $A \in (f, c)_{reg}$ if $Ax \in c$ for $x \in f$ with $\lim Ax = f - \lim x$. The following are the necessary and sufficient conditions for A to be strongly regular (cf. Lorentz [35]):

Theorem 2.1.7. $A \in (f, c)_{reg}$ if and only if A is regular and

(2.1.7.1) $\lim_n \sum_k \mid a_{nk} - a_{n,k+1} \mid = 0$.

The matrix A is said to be f-regular, i.e. $A \in (f, f)_{reg}$ if $Ax \in f$ for $x \in f$ with $f - \lim Ax = f - \lim x$. The following are the necessary and sufficient conditions for A to be f-regular (cf. Duran [12]):

Theorem 2.1.8. $A \in (f, f)_{reg}$ if and only if A is almost regular and

(2.1.8.1) $\lim_p \sum_k \frac{1}{p+1} \mid \sum_{i=0}^{p} (a_{n+i,k} - a_{n+i,k+1}) \mid = 0$ uniformly in n.

Lemma 2.1.9 (Eizen and Laush [13]). $A \in (\ell_\infty, f)$ if and only if

(2.1.9.1) $\sup_n \sum_k \mid \frac{1}{p+1} \sum_{i=0}^{p} a_{n+i,k} \mid < \infty;$

(2.1.9.2) $\lim_p \frac{1}{p+1} \sum_{i=0}^{p} a_{n+i,k} = \beta_k$ uniformly in n for each k;

(2.1.9.3) $\lim_p \sum_k \mid \frac{1}{p+1} \sum_{i=0}^{p} (a_{n+i,k} - \beta_k) \mid = 0$ uniformly in n.

In this chapter we characterize statistically strongly regular matrices and statistically almost regular matrices analogous to Lorentz [35] and King [27].

2.2. Some Matrix Classes Involving the Space st

We need the following results which will be used in establishing our main theorems.

Lemma 2.2.1 (Kolk [31]). Let X be a sequence space. Then $A \in (X, st)$ if and only if for every $x \in X$ there exists an index set N such that $\delta(N) = 1$ and $A^{[N]}x \in c$, where $A^{[N]} = (d_{nk})$ for all $k \in \mathbb{N}$,

$$d_{nk} = \begin{cases} a_{nk} & , \text{ if } n \in N, \\ 0 & , \text{ otherwise;} \end{cases}$$

is called the N-section of $A = (a_{nk})$.

By $A \in (c, st \cap \ell_\infty)_{reg}$ *we mean that* $Ax \in st \cap \ell_\infty$ *for all* $x \in c$ *with* $st - \lim Ax = \lim x$.

We will call such matrices as <u>statistically left regular</u> matrices.

Theorem 2.2.2 (Kolk [31]). $A \in (c, st \cap \ell_\infty)_{reg}$ if and only if

(2.2.2.1) $\|A\| < \infty,$

and there exists an index set $N = \{n_i\}$ such that $\delta(N) = 1$ and

(2.2.2.2) $\lim_i a_{n_ik} = 0 \quad (k \in \mathbb{N})$,

(2.2.2.3) $\lim_i \sum_k a_{n_ik} = 1$.

By $A \in (st \cap \ell_\infty, c)_{reg}$ we mean that $Ax \in c$ for all $x \in st \cap \ell_\infty$ with $\lim Ax = st - \lim x$.

We will call such matrices as statistically right regular matrices.

Theorem 2.2.3 (Maddox[37], Kolk [30]). $A \in (w_p \cap \ell_\infty, c)_{reg} = (\dot{s}t \cap \ell_\infty, c)_{reg}$ if and only if A is regular and

(2.2.3.1) $\lim_n \sum_{k \in E} \mid a_{nk} \mid = 0$ for every $E \subseteq \mathbb{N}$ such that $\delta(E) = 0$.

By $A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$ we mean that $Ax \in st \cap \ell_\infty$ for all $x \in st \cap \ell_\infty$ with $st - \lim Ax = st - \lim x$.

We will call such matrices as statistically regular matrices.

Connor [6] introduced the following:

Let $s = (s_i)$ be a strictly increasing sequence of integers with $1 \leq s_i$. We say that $s \in \mathrm{S}$ if

$$\lim_m (s_{2m})^{-1} \sum_{\ell=1}^{m} (s_{2\ell} - s_{2\ell-1}) = 0.$$

Theorem 2.2.4. Let $A = (a_{nk})$ be a matrix. The matrix A maps bounded statistically null sequences into null sequences if and only if A maps null sequences into null sequences and

$$\lim_n \sum_{\ell=1}^{\infty} \sum_{k=s_{2\ell-1}}^{s_{2\ell}} \mid a_{nk} \mid = 0$$

for every $s \in \mathrm{S}$.

2.3. Main Results

First we define the following new matrix classes.

Definition 2.3.1. An infinite matrix $A = (a_{nk})$ is said to be statistically almost regular if $Ax \in f$ for all $x \in st \cap \ell_\infty$ with $st - \lim x = f - \lim Ax$, i.e. $A \in (st \cap \ell_\infty, f)_{reg}$.

Definition 2.3.2. An infinite matrix $A = (a_{nk})$ is said to be statistically strongly regular if $Ax \in st \cap \ell_\infty$ for all $x \in f$ with $f - \lim x = st - \lim Ax$, i.e. $A \in (f, st \cap \ell_\infty)_{reg}$.

For typographical convenience we write

$$t(n,k,p) = \frac{1}{p+1}\sum_{i=0}^{p} a_{n+i,k}.$$

Now we characterize these classes :

Theorem 2.3.3. $A \in (st \cap \ell_\infty, f)_{reg}$ if and only if A is almost regular and

(2.3.3.1) $\lim_p \sum_{k \in E} \mid t(n,k,p) \mid = 0$ uniformly in n for every $E \subseteq \mathbb{N}$

such that $\delta(E) = 0$.

Proof. Necessity. Let $A \in (st \cap \ell_\infty, f)_{reg}$ and $st - \lim x = f - \lim Ax = \ell$, say. Since $c \subset st$, we have $A \in (c, f)_{reg}$, i.e. A is almost regular.

Let $E \subseteq \mathbb{N}$ with $\delta(E) = 0$ and let $x \in \ell_\infty$. Then the E- section y of x converges statistically to zero, and $y \in \ell_\infty$. Hence $y \in st \cap \ell_\infty$ and so $Ay \in f$ with $st - \lim y = 0 = f - \lim Ay$. Also

$$A_n^{[E]}(x) = A_n(y), \quad n = 1, 2, \cdots ,$$

which implies that $A^{[E]}(x) = (A_n^{[E]}(x))_{n=1}^{\infty} \in f$ and $f - \lim A^{[E]}(x) = 0$. Then $A^{[E]} \in (\ell_\infty, f)$ for every index set E with $\delta(E) = 0$ and so by condition (2.1.9.3) with $\beta_k = 0$ we must have (2.3.3.1).

Sufficiency. Let $x \in st \cap \ell_\infty$ with $st - \lim x = \ell$, say. There are two cases:

(a) If $x \in c \subset st$, then $Ax \in f$ with $f - \lim Ax = \lim x$ $(= st - \lim x = \ell)$. Hence $A \in (st \cap \ell_\infty, f)_{reg}$.

(b) If $x \in st \backslash c$ then by Theorem 1.4.6,

$$x = y + z,$$

where $y \in c$ and $z \in st_o$ with $st - \lim x = \lim y$. That is there exists an index set E with $\delta(E) = 0$ such that $\lim y = st - \lim x = \ell$, where $y = (y_k)$ is the $\mathbb{N} \backslash E-$section of x. We can write

$$Ax = Ay + Az.$$

Since $z \in st_o$ implies that $\lim Az = 0$ by Theorem 2.2.4 and so by (2.3.3.1) $f - \lim Az = 0$. Therefore

$$f - \lim Ax = f - \lim Ay.$$

Further as A is almost regular $Ay \in f$ for $y \in c$ with $f - \lim Ay = \lim y$. Hence

$$f - \lim Ax = \lim y = st - \lim x = \ell,$$

i.e. $A \in (st \cap \ell_\infty, f)_{reg}$.

This completes the proof of the theorem.

Theorem 2.3.4. $A \in (f, st \cap \ell_\infty)_{reg}$ if and only if

(2.3.4.1) A is statistically left regular, i.e. $A \in (c, st \cap \ell_\infty)_{reg}$;

(2.3.4.2) there exists an index set $N = \{n_i\}$ such that $\delta(N) = 1$ and

$$\lim_i \sum_k \mid a_{n_i k} - a_{n_i, k+1} \mid = 0.$$

Proof. Necessity. Condition (2.3.4.1) follows easily from the fact that

$$(f, st \cap \ell_\infty)_{reg} \subseteq (c, st \cap \ell_\infty)_{reg}.$$

Let (2.3.4.2) do not hold. Then obviously

$$\lim_{n} \sum_{k} \mid a_{nk} - a_{n,k+1} \mid \neq 0.$$

Hence $A \notin (f,c)_{reg} \subseteq (f, st \cap \ell_\infty)_{reg}$, i.e. $A \notin (f, st \cap \ell_\infty)_{reg}$.
Therefore a contradiction and so (2.3.4.2) must hold.

<u>Sufficiency</u>. Let the conditions hold and $x \in f$ with $f - \lim x = L$, say. Now

$$\text{(2.3.4.3)} \qquad \sum_{k=0}^{\infty} a_{n_i k} x_k = \sum_{k=0}^{\infty} a_{n_i k}\Big(\frac{1}{p+1}\sum_{r=k}^{k+p} x_r\Big)$$

$$- \sum_{k=p}^{\infty}\Big(\frac{a_{n_i k} + \cdots + a_{n_i,k-p}}{p+1} - a_{n_i k}\Big)x_k$$

$$+ \sum_{k=0}^{p-1} a_{n_i k} x_k + \sum_{k=0}^{p-1}\Big(\frac{a_{n_i k} + \cdots + a_{n_i,k-p+1}}{p+1}\Big)x_k.$$

From the condition (2.3.4.1) and Theorem 2.2.2, it follows that the third and fourth summations in (2.3.4.3) tend to zero as $i \to \infty$.

Write

$$D_{ip} = \sum_{k=p}^{\infty}\Big(\frac{a_{n_i k} + \cdots + a_{n_i,k-p}}{p+1} - a_{n_i k}\Big)x_k.$$

Then

$$\mid D_{ip} \mid \leq \frac{1}{p+1}\sum_{k=p}^{\infty} \mid a_{n_i k} + \cdots + a_{n_i,k-p} - (p+1)a_{n_i k} \mid \mid x_k \mid$$

$$\leq \frac{\|x\|}{p+1}\sum_{r=0}^{p} \mid a_{n_i,k-r} - a_{n_i k} \mid$$

$$\leq \frac{\|x\|}{p+1}\sum_{r=0}^{p} r \sum_{k=0}^{\infty} \mid a_{n_i k} - a_{n_i,k+1} \mid$$

$$\leq \frac{p}{2}\ \|x\| \sum_{k=0}^{\infty} \mid a_{n_ik} - a_{n_i,k+1} \mid .$$

$$\rightarrow 0 \quad \text{as} \quad i \rightarrow \infty \qquad\qquad \text{by (2.3.4.2).}$$

Finally (2.3.4.3) reduces to

$$\lim_i \sum_{k=0}^{\infty} a_{n_ik} x_k = \lim_i \sum_{k=0}^{\infty} a_{n_ik} \left(\frac{1}{p+1} \sum_{r=0}^{p} x_{k+r} \right)$$

$$= L \ \lim_i \sum_{k=0}^{\infty} a_{n_ik}, \quad \text{since} \ \ f - \lim x = L;$$

$$= L \quad \text{by condition (2.2.2.3)} .$$

Hence

$$st - \lim Ax = L = f - \lim x,$$

i.e. $A \in (f, st \cap \ell_\infty)_{reg}$.

This completes the proof of the theorem.

CHAPTER III

SOME STATISTICAL CORE THEOREMS

3.1. Introduction

The Knopp core (or K-core) of a real bounded sequence x is defined to be the closed interval $[\ell(x), L(x)]$, where

$$\ell(x) = \liminf x; \quad L(x) = \limsup x \ .$$

The well-known Knopp's core theorem states that (cf. Knopp [28], Maddox [38]) :

In order that $L(Ax) \leq L(x)$ for every real bounded sequence x, it is necessary and sufficient that A should be regular and $\lim_n \sum_k \mid a_{nk} \mid = 1$.

Note that $L(Ax) \leq L(x)$ means $K - \text{core}\{Ax\} \subseteq K - \text{core}\{x\}$.

Shcherbakov [52] has shown that for every bounded complex sequence x,

$$K - \text{core}\{x\} = \bigcap_{z \in \mathbb{C}} K_x^*(z),$$

where

$$K_x^*(z) := \{w \in \mathbb{C} : \mid w - z \mid \leq \limsup_k \mid x_k - z \mid\} \ .$$

If x is a statistically bounded sequence, then the statistical core of x (cf. Friday and Orhan [22]) is defined to be the closed interval $[st - \liminf x,\ st - \limsup x]$.

It is noted that

$$\liminf x \leq st - \liminf x \leq st - \limsup x \leq \limsup x$$

and consequently
$$st-\text{core}\{x\}\subseteq K-\text{core}\{x\}\ .$$

Fridy and Orhan [23] introduced and studied the equivalent form of statistical core and proved that
$$st-\text{core}\{x\}=\bigcap_{z\in\mathbb{C}} S_x^*(z)\ ,$$
where
$$S_x^*(z):=\{w\in\mathbb{C}:\mid w-z\mid\leq st-\limsup_k\mid x_k-z\mid\}$$
for a statistically bounded complex sequence x.

Analogous to the Knopp core theorem, in [23] necessary and sufficient conditions were established for
$$K-\text{core}\{Ax\}\subseteq st-\text{core}\{x\}$$
for every bounded complex sequence x.

The sufficient conditions were also derived for A to yield
$$st-\text{core}\{Ax\}\subseteq st-\text{core}\{x\}.$$

Recently Li and Fridy [34] obtained the necessary and sufficient conditions for A to yield
$$st-\text{core}\{Ax\}\subseteq st-\text{core}\{x\},$$
and moreover
$$st-\text{core}\{Ax\}\subseteq K-\text{core}\{x\}$$
through the concepts of statistical partition and superior partition of $\mathbb{N}$.

In the present chapter we also consider the same inclusions as in [34] and obtain necessary and sufficient conditions in a more natural way by using the matrix classes involving the space of statistically convergent sequences. Our conditions are stronger than that of Li and Fridy and proofs are easier and shorter.

Also we obtain necessary and sufficient conditions to establish the inclusions
$$st-\text{core}\{Ax\}\subseteq K-\text{core}\{Tx\},$$

(3.2.3.4) $\lim_n \sum_{k=1}^{\infty} | c_{nk} | = 1$;

(3.2.3.5) for any fixed n,

$$\lim_m \sum_{k=0}^{m} | \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} | = 0.$$

Theorem 3.2.4 (Fridy and Orhan [23]). Let $T = (t_{nk})$ be a normal matrix and denote its triangular inverse by $T^{-1} = (t_{nk}^{-1})$. For an arbitrary matrix A, in order that, whenever $Tx \in \ell_\infty$, Ax should exist and be bounded and satisfy

(3.2.4.1) $$K - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\}$$

it is necessary and sufficient that the following conditions hold:

(3.2.4.2) $C = (c_{nk}) = AT^{-1}$ exists;

(3.2.4.3) C is regular and $\lim_n \sum_{k \in E} | c_{nk} | = 0$, whenever $\delta(E) = 0$ for $E \subseteq \mathbb{N}$;

(3.2.4.4) $\lim_n \sum_{k=1}^{\infty} | c_{nk} | = 1$;

(3.2.4.5) for any fixed n,

$$\lim_m \sum_{k=0}^{m} | \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} | = 0.$$

Theorem 3.2.5 (Fridy and Orhan [23]). If A and T satisfy conditions (3.2.4.2)-(3.2.4.5), then

(3.2.5.1) $$st - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\}$$

for every x such that $Tx \in \ell_\infty$. But converse need not be true in general.

Theorem 3.2.6 (Li and Fridy [34]). If A is a matrix for which $\{\sum_{j=1}^{\infty} | a_{nj} |\}_{n=1}^{\infty}$ is statistically bounded, then

(3.2.6.1) $$st - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}$$

for every $x \in \ell_\infty$ if and only if

(3.2.6.2) $st-\lim_n \sum_{j\in E} a_{nj} = 1$, for every $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$;

(3.2.6.3) $st-\limsup_n \sum_{i=1}^{\ell} \mid \sum_{j\in K_i} a_{nj} \mid \leq 1$, whenever $\{K_1, K_2, \cdots, K_\ell\}$ is a st-partition of $\mathbb{N}$.

Theorem 3.2.7 (Li and Fridy [34]). If A is a matrix for which $\{\sum_{j=1}^{\infty} \mid a_{nj} \mid\}_{n=1}^{\infty}$ is statistically bounded, then

(3.2.7.1) $$st-\text{core}\{Ax\} \subseteq K-\text{core}\{x\}$$

for every $x \in \ell_\infty$ if and only if

(3.2.7.2) $st-\lim_n \sum_{j\in E} a_{nj} = 1$, whenever $\mathbb{N}\backslash E$ is finite;

(3.2.7.3) $st-\limsup_n \sum_{i=1}^{\ell} \mid \sum_{j\in K_i} a_{nj} \mid \leq 1$, whenever $\{K_1, K_2, \cdots, K_\ell\}$ is a sup-partition of $\mathbb{N}$.

3.3. Main Results

In this section we give alternative conditions for the above core Theorems 3.2.6 and 3.2.7.

Theorem 3.3.1. If $\|A\| < \infty$, then for every $x \in \ell_\infty$

(3.3.1.1) $$st-\text{core}\{Ax\} \subseteq K-\text{core}\{x\}$$

if and only if

(3.3.1.2) $st-\lim_n \sum_{k\in E} a_{nk} = 1$, whenever $\mathbb{N}\backslash E$ is finite, where $E \subseteq \mathbb{N}$;

(3.3.1.3) $A \in (c, st \cap \ell_\infty)_{reg}$.

Proof . Necessity. Suppose that

$$st-\text{core}\{Ax\} \subseteq K-\text{core}\{x\}$$

and $x \in \ell_\infty$ has a limit point ℓ. Then

$$\{\ell\} = K - \text{core}\{x\} \supseteq st - \text{core}\{Ax\}.$$

Since $\|A\| < \infty$, $Ax \in \ell_\infty$ for $x \in \ell_\infty$ by Lemma 2.1.3. Hence

$$\liminf x \leq st - \liminf Ax \leq st - \limsup Ax \leq \limsup x.$$

But $\liminf x = \limsup x = \ell$, so that

$$st - \liminf Ax = st - \limsup Ax = \ell.$$

That is $st - \lim Ax = \lim x = \ell$ and so $st - \text{core}\{Ax\} = \{\ell\}$. Hence $A \in (c, st \cap \ell_\infty)_{reg}$, i.e. condition (3.3.1.3).

To prove (3.3.1.2), let us define $x = (x_k)$ by

$$x_k = \begin{cases} 1 & , \text{ if } k \in E, \\ 0 & , \text{ otherwise;} \end{cases}$$

where $E \subseteq \mathbb{N}$ such that $\mathbb{N} \backslash E$ is finite. Then

$$K - \text{core}\{x\} = \{1\}.$$

Since $\|A\| < \infty$ implies $Ax \in \ell_\infty$ for $x \in \ell_\infty$, Ax has at least one statistical cluster point by Proposition 1.4.1. Now, by proposition 1.4.2, the set of statistical cluster points is in $st - \text{core}\{Ax\}$. Therefore, $st - \text{core}\{Ax\} \neq \emptyset$. Since

$$st - \text{core}\{Ax\} \subseteq K - \text{core}\{x\} = \{1\},$$

we have $st - \text{core}\{Ax\} = \{1\}$ and 1 is the only statistical cluster point of Ax. Using proposition 1.4.4, we have $st - \lim Ax = 1$, i.e.

$$st - \lim_n \sum_{k \in E} a_{nk} = 1, \text{ whenever } \mathbb{N} \backslash E \text{ is finite.}$$

Hence (3.3.1.2) holds.

Sufficiency. Let conditions (3.3.1.2) and (3.3.1.3) hold and that $w \in st - \text{core}\{Ax\}$. Then for any $z \in \mathbb{C}$, we have

$$\begin{aligned}
| w - z | &\leq st - \limsup_n | z - A_n(x) | \\
&= st - \limsup_n | z - \sum_{k=1}^{\infty} a_{nk} x_k | \\
&\leq st - \limsup_n | \sum_{k=1}^{\infty} a_{nk}(z - x_k) | \\
&\quad + st - \limsup_n | z | \, | 1 - \sum_{k=1}^{\infty} a_{nk} |
\end{aligned}$$

$$(3.3.1.4) \qquad = st - \limsup_n | \sum_{k=1}^{\infty} a_{nk}(z - x_k) | \quad , \quad \text{by (3.3.1.2).}$$

Let $r = \limsup_k | z - x_k |$ and $E := \{k : | z - x_k | > r + \varepsilon\}$ for $\varepsilon > 0$. Then $\delta(E) = 0$ as E is finite and we have

$$(3.3.1.5) \qquad | \sum_k a_{nk}(z - x_k) | \leq \sup_k | z - x_k | \, | \sum_{k \in E} a_{nk} | + (r + \varepsilon) \, | \sum_{k \notin E} a_{nk} | \, .$$

Therefore, by conditions (3.3.1.2) and (3.3.1.3), we obtain

$$st - \limsup_n | \sum_{k=1}^{\infty} a_{nk}(z - x_k) | \leq r + \varepsilon.$$

Hence by (3.3.1.4) we have

$$| w - z | \leq r + \varepsilon$$

and since ε is arbitrary,

$$| w - z | \leq r = \limsup_k | z - x_k |,$$

i.e. $w \in K_x^*(z)$. Hence $w \in K - \text{core}\{x\}$ and so

$$st - \text{core}\{Ax\} \subseteq K - \text{core}\{x\}.$$

This completes the proof of the theorem.

Remark 3.3.2. Condition (3.3.1.2) can not be replaced by

$$(3.3.2.1) \qquad st-\lim_n \sum_{k\in E} a_{nk} = 1, \text{ for any set } E \subseteq \mathbb{N} \text{ such that } \delta(E) = 1.$$

Consider the following example.

Example 3.3.3. Let $A = (a_{nk})$ be an infinite matrix defined as

$$a_{nk} = \begin{cases} 1 & , \text{ if } n \text{ is a nonsquare and } k = n^2, \\ 0 & , \text{ otherwise.} \end{cases}$$

Then

$$\sum_k a_{nk} = \begin{cases} 1 & , \text{ if } n \text{ is a nonsquare,} \\ 0 & , \text{ otherwise.} \end{cases}$$

We see that $A \in (c, st \cap \ell_\infty)_{reg}$ but A is not regular. Further, for any set $E \subseteq \mathbb{N}$ such that $\mathbb{N}\backslash E$ is finite we have

$$st-\lim_n \sum_{k\in E} a_{nk} = 1.$$

So that for any bounded sequence x we have

$$st-\text{core}\{Ax\} \subseteq K-\text{core}\{x\}.$$

Now, let $E = \{k \neq n^2 : k \in \mathbb{N}\}$. Then $\delta(E) = 1$ and we have

$$\sum_{k\in E} a_{nk} = 0, \quad \text{for all } n.$$

Hence

$$st-\lim_n \sum_{k\in E} a_{nk} = 0.$$

Further, for any bounded sequence, e.g. $x = (1, 1, \cdots)$, we have $K-\text{core}\{x\} = \{1\}$ and

$$\sum_k a_{nk}x_k = \begin{cases} 1 & , \text{ if } n \text{ is a nonsquare,} \\ 0 & , \text{ otherwise.} \end{cases}$$

So that $st-\lim Ax = 1$, i.e.

$$st-\text{core}\{Ax\} = \{1\} = K-\text{core}\{x\}.$$

Therefore, we see that (3.3.1.2) hold but (3.3.2.1) does not hold. Hence condition (3.3.1.2) is necessary.

Theorem 3.3.4. If $\|A\| < \infty$, then for every $x \in \ell_\infty$

(3.3.4.1) $$st-\text{core}\{Ax\} \subseteq st-\text{core}\{x\}$$

if and only if

(3.3.4.2) $st-\lim_n \sum_{k\in E} a_{nk} = 1$, for every $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$;

(3.3.4.3) $A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$.

Proof . Necessity. Let $st-\text{core}\{Ax\} \subseteq st-\text{core}\{x\}$ and x be statistically convergent to the number ℓ. Then

$$\{\ell\} = st-\text{core}\{x\} \supseteq st-\text{core}\{Ax\}.$$

Since $\|A\| < \infty$ implies $Ax \in \ell_\infty$ for $x \in \ell_\infty$, we have

$$st-\liminf x \leq st-\liminf Ax \leq st-\limsup Ax \leq st-\limsup x.$$

But $st-\liminf x = st-\limsup x = \ell$, so that

$$st-\lim Ax = st-\lim x = \ell,$$

i.e. $st-\text{core}\{Ax\} = \{\ell\}$. Hence $A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$.

To prove (3.3.4.2), let $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$. Let χ_E be the characteristic function of E. Then

$$st-\text{core}\{\chi_E\} = \{1\}.$$

Since $\|A\| < \infty$ implies $A\chi_E \in \ell_\infty$ for $\chi_E \in \ell_\infty$, we have that $A\chi_E$ has at least one statistical cluster point. Therefore, $st - \text{core}\{A\chi_E\} \neq \emptyset$. Also $st - \text{core}\{A\chi_E\} = \{1\}$, since

$$st - \text{core}\{A\chi_E\} \subseteq st - \text{core}\{\chi_E\} = \{1\}.$$

Hence

$$st - \lim A\chi_E = st - \lim_n \sum_{k \in E} a_{nk} = 1, \text{ where } \delta(E) = 1,$$

i.e. condition (3.3.4.2).

Sufficiency. It follows on the same lines as in Theorem 3.3.1, i.e. for $w \in st - \text{core}\{Ax\}$, we arrived at

$$| w - z | \leq r, \text{ where } r = st - \limsup_k | z - x_k |, \text{ for any } z \in \mathbb{C},$$

by using conditions (3.3.4.2) and (3.3.4.3). So that $w \in S_x^*(z)$. Hence $w \in st - \text{core}\{x\}$, i.e.

$$st - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}.$$

This completes the proof of the theorem.

Remark 3.3.5. If $st - \lim_n \sum_{k \in E} a_{nk} = 1$, where $\delta(E) = 1$ then

$$st - \limsup_n | \sum_{k \in E} a_{nk} | = 0, \quad \text{whenever} \quad \delta(E) = 0.$$

Similarly, if $st - \lim_n \sum_{k \in E} a_{nk} = 1$, whenever $\mathbb{N} \setminus E$ is finite then

$$st - \lim_n \sum_{k \in E} a_{nk} = 0. \quad \text{whenever} \quad E \text{ is finite}.$$

Remark 3.3.6. Note that relaxing the condition on the matrix A we get stronger necessary condition than condition (3.2.6.3) (i.e. Theorem 3 of Li and Fridy [34]). To show this we prove the following proposition.

Proposition 3.3.7. Let $A = (a_{nk})$ be an infinite matrix such that $\|A\| < \infty$. If

(3.3.7.1) $\quad A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$,

then

(3.3.7.2) $\quad st - \limsup_n \sum_{i=1}^{\ell} \mid \sum_{j \in K_i} a_{nj} \mid \leq 1$,

whenever $\{K_1, K_2, \cdots, K_\ell\}$ is a st-partition of $\mathbb{N}$,

but not conversely.

Proof. Let $A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$ and suppose that (3.3.7.2) does not hold. Then

$$st - \limsup_n \sum_{i=1}^{\ell} \mid \sum_{j \in K_i} a_{nj} \mid = \beta > 1,$$

where $\{K_1, K_2, \cdots, K_\ell\}$ is a st-partition of $\mathbb{N}$.

Since A is bounded which implies that it is statistically bounded. Then by Theorem 1.4.3 we can find ℓ real numbers $\alpha_1, \alpha_2, \cdots, \alpha_\ell$ such that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = \beta, \tag{3.3.7.3}$$

and

$$\delta^*\{n \in \mathbb{N} : \sum_{i=1}^{\ell} \mid \sum_{j \in K_i} a_{nj} - \alpha_i \mid < \varepsilon\} > 0. \tag{3.3.7.4}$$

Let $K = \bigcup_{i=1}^{\ell} K_i$ and let us define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1 & , \text{ if } k \in K, \\ 0 & , \text{ otherwise .} \end{cases}$$

It is easy to see that $st - \lim x = 1$, since $\delta(K) = 1$.

Now

$$| \sum_{j=1}^{\infty} a_{nj}x_j | = | \sum_{i=1}^{\ell}\Big(\sum_{j\in K_i} a_{nj}x_j \Big) + \sum_{j\notin K} a_{nj}x_j |$$

$$= | \sum_{i=1}^{\ell}\Big(\sum_{j\in K_i} a_{nj} - \alpha_i \Big) + \sum_{i=1}^{\ell} \alpha_i |$$

$$\geq | \sum_{i=1}^{\ell} \alpha_i | - | \sum_{i=1}^{\ell}\Big(\sum_{j\in K_i} a_{nj} - \alpha_i \Big) | .$$

$$> \beta - \varepsilon \qquad \text{by (3.3.7.3) and (3.3.7.4).}$$

Now

$$F = \{n \in \mathbb{N} : \sum_{i=1}^{\ell} | \sum_{j\in K_i} a_{nj} - \alpha_i | < \varepsilon\} \subseteq \{n \in \mathbb{N} : | \sum_{j=1}^{\infty} a_{nj}x_j | > \beta - \varepsilon\} = G.$$

Therefore

$$\delta^*(G) \geq \delta^*(F) > 0.$$

This implies that

$$st - \limsup Ax \geq \beta > 1 = st - \lim x$$

which contradicts our hypothesis that $A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$.

Hence (3.3.7.2) must hold.

For the converse part see Example 2 of Li and Fridy [34]. For the sake of completeness we would like to include it here.

Define the matrix A by

$$a_{nk} = \begin{cases} 1 & , \text{ if } n = k, \\ \frac{1}{2} & , \text{ if } k \text{ is a square and } (\sqrt{k} - 1)^2 \leq n < k, \\ 0 & , \text{ otherwise;} \end{cases}$$

and $x = (x_k)$ defined by

$$x_k = \begin{cases} 2 & , \text{ if } k \text{ is a square,} \\ 1 & , \text{ otherwise .} \end{cases}$$

Then $st - \lim x = 1$, $st - \lim Ax = \frac{3}{2}$ and condition (3.3.7.2) holds but $A \notin (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$.

Similarly we can show that

Proposition 3.3.8. Let $A = (a_{nk})$ be an infinite matrix such that $\|A\| < \infty$. If

(3.3.8.1) $A \in (c, st \cap \ell_\infty)_{reg}$,

then

(3.3.8.2) $st - \limsup_n \sum_{i=1}^{\ell} | \sum_{j \in K_i} a_{nj} | \leq 1$,

whenever $\{K_1, K_2, \cdots, K_\ell\}$ is a sup-partition of $\mathbb{N}$,

but not conversely.

Theorem 3.3.9. Let $T = (t_{jk})$ be a normal matrix and $A = (a_{nj})$ be any matrix. In order that whenever Tx is bounded Ax should exist and be bounded and satisfy

(3.3.9.1) $$st - \text{core}\{Ax\} \subseteq K - \text{core}\{Tx\}$$

it is necessary and sufficient that

(3.3.9.2) $(c_{nk}) = C = AT^{-1}$ exists;

(3.3.9.3) $C \in (c, st \cap \ell_\infty)_{reg}$;

(3.3.9.4) $st - \lim_n \sum_{k \in E} c_{nk} = 1$, whenever $\mathbb{N} \backslash E$ is finite;

(3.3.9.5) for any fixed n,

$$\sum_{k=0}^{m} | \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} | \to 0 \text{ as } m \to \infty.$$

Proof. Necessity. Let us suppose that $A_n(x)$ exist for every n, whenever $Tx \in \ell_\infty$. Then by Lemma 2.1.5, it follows that conditions (3.3.9.2) and (3.3.9.5) hold, and $Ax = Cy$, where $y = Tx$. Since $Ax \in \ell_\infty$, (3.3.9.1) implies that

(3.3.9.6) $$st - \text{core}\{Cy\} \subseteq K - \text{core}\{y\}.$$

Now, by Theorem 3.3.1, (3.3.9.6) implies that $C \in (c, st \cap \ell_\infty)_{reg}$ and $st - \lim_n \sum_{k \in E} c_{nk} = 1$, whenever $\mathbb{N} \backslash E$ is finite, i.e. conditions (3.3.9.3) and (3.3.9.4) hold.

Sufficiency. Let the conditions (3.3.9.2)-(3.3.9.5) hold. Then by Theorem 3.3.1, $C \in (c, st \cap \ell_\infty)_{reg}$. Therefore by Theorem 2.2.2, $\sup_n \sum_k \mid c_{nk} \mid < \infty$ which implies that $\sum_k \mid c_{nk} \mid < \infty$ holds for every n. So that by Lemma 2.1.5, the conditions of Theorem 3.2.3 are satisfied and consequently

$$A_n(x) = \sum_k c_{nk} y_k$$

exists. Further use of Lemma 2.1.5 yields $Ax = Cy \in \ell_\infty$, where $y = Tx$ which implies by Theorem 3.3.1 that

$$st - \text{core}\{Cy\} \subseteq K - \text{core}\{y\}.$$

Hence (3.3.9.1) holds.

This completes the proof of the theorem.

On the same lines by using Theorem 3.3.4, we can easily prove the following:

Theorem 3.3.10. Let A and T be same as in Theorem 3.3.9. Then

(3.3.10.1) $$st - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\}$$

if and only if (3.3.9.2) and (3.3.9.5) hold and

(3.3.10.2) $C \in (st \cap \ell_\infty, st \cap \ell_\infty)_{reg}$;

(3.3.10.3) $st - \lim_n \sum_{k \in E} c_{nk} = 1$, for every $E \subseteq \mathbb{N}$ such that $\delta(E) = 1$.

CHAPTER IV

BANACH CORE AND RELATED INCLUSIONS

4.1. Introduction

Banach core of a sequence, analogous to the Knopp core, is inherently connected to the concept of Banach limit. The Banach core (or B-core) of a real bounded sequence x is defined to be the closed interval $[-q(-x), q(x)]$, where

$$q(x) = \lim_p \sup_n \sup t_{pn}(x)$$

is a sublinear functional on ℓ_∞.

Like Shcherbakov [52], it is natural to extend this definition for B-core, i.e. for every complex bounded sequence x

$$B - \text{core}\{x\} = \bigcap_{z \in \mathbb{C}} B_x^*(z),$$

where

$$B_x^*(z) := \{w \in \mathbb{C} : \mid w - z \mid \leq \lim_p \sup_n \sup \mid t_{pn}(x) - z \mid\}.$$

Note that $q(x) \leq L(x)$ for all $x \in \ell_\infty$, where $L(x) = \limsup x$. Hence it follows that

$$B - \text{core}\{x\} \subseteq K - \text{core}\{x\}.$$

From Section 1.6, we note that

$$B - \text{core}\{x\} \subseteq st - \text{core}\{x\}, \quad \text{for all } \ x \in \ell_\infty\ .$$

In this chapter we determine necessary and sufficient conditions for the inclusions

$$B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\},$$

and

$$st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\},$$

further, we extend these results to the following inclusions

$$B - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\},$$

and

$$st - \text{core}\{Ax\} \subseteq B - \text{core}\{Tx\},$$

where T is a normal matrix, and also we prove some core theorems analogous to Orhan [44].

4.2. Some Previous Results

Analogues and extensions of Knopp core theorem have been established by various authors (cf. Choudhary [4], Das [10], Maddox [38], Mursaleen [41]). In [44] Orhan has proved the following :

Theorem 4.2.1. $B - \text{core}\{Ax\} \subseteq K - \text{core}\{x\}$ if and only if A is almost regular and

(4.2.1.1) $\limsup_p \sup_n \sum_k \mid \frac{1}{p+1} \sum_{i=0}^{p} a_{n+i,k} \mid = 1$.

Theorem 4.2.2. $B - \text{core}\{Ax\} \subseteq B - \text{core}\{x\}$ if and only if A is f-regular and (4.2.1.1) holds.

4.3. Main Results

Theorem 4.3.1. If $\|A\| < \infty$, then for every $x \in \ell_\infty$

$$B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\} \tag{4.3.1.1}$$

if and only if

(4.3.1.2) A is almost regular, and

$$\lim_p \sum_{k \in E} \mid t(n,k,p) \mid = 0 \text{ uniformly in } n, \text{ whenever } \delta(E) = 0 \text{ for } E \subseteq \mathbb{N};$$

(4.3.1.3) $\limsup_p \sup_n \sum_k \mid t(n,k,p) \mid = 1.$

Proof . Necessity. Let $B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}$ and x be statistically convergent to the number ℓ. Then

$$\{\ell\} = st - \text{core}\{x\} \supseteq B - \text{core}\{Ax\},$$

i.e.
$$q(Ax) \leq st - \lim x = \ell.$$

Since $\|A\| < \infty$ if and only if $A \in (\ell_\infty, \ell_\infty)$ by Lemma 2.1.3, i.e. $Ax \in \ell_\infty$ for $x \in \ell_\infty$. Hence

$$st - \liminf x \leq -q(-Ax) \leq q(Ax) \leq st - \limsup x.$$

But $st - \liminf x = st - \limsup x = \ell$. So that

$$q(Ax) = -q(-Ax) = \ell,$$

i.e. $B - \text{core}\{Ax\} = \{\ell\}$. This implies that $A \in (st \cap \ell_\infty, f)_{reg}$ and hence by Theorem 2.2.3 we have that A is almost regular and

$$\lim_p \sum_{k \in E} \mid t(n,k,p) \mid = 0 \text{ uniformly in } n, \text{ whenever } \delta(E) = 0,$$

i.e. condition (4.3.1.2).

Also, we know that $st - \text{core}\{x\} \subseteq K - \text{core}\{x\}$. Hence

$$B - \text{core}\{Ax\} \subseteq K - \text{core}\{x\}$$

and Theorem 4.2.1 gives that

$$\lim_p \sup \sup_n \sum_k \mid t(n,k,p) \mid = 1,$$

i.e. condition (4.3.1.3).

Sufficiency. Let conditions (4.3.1.2) and (4.3.1.3) hold and that $w \in B - \text{core}\{Ax\}$. Then for any $z \in \mathbb{C}$ we have

$$\begin{aligned} \mid w - z \mid &\leq \lim_p \sup \sup_n \mid z - t_{pn}(Ax) \mid \\ &\leq \lim_p \sup \sup_n \mid z - \sum_k t(n,k,p) x_k \mid \\ &\leq \lim_p \sup \sup_n \mid \sum_k t(n,k,p)(z - x_k) \mid \\ &+ \lim_p \sup \sup_n \mid z \mid \mid 1 - \sum_k t(n,k,p) \mid \\ &= \lim_p \sup \sup_n \mid \sum_k t(n,k,p)(z - x_k) \mid \text{ , by (4.3.1.3).} \end{aligned}$$

Hence

$$\mid w - z \mid \leq \lim_p \sup \sup_n \mid \sum_k t(n,k,p)(z - x_k) \mid . \tag{4.3.1.4}$$

Let $r = st - \lim\sup_k \mid z - x_k \mid$ and $E := \{k : \mid z - x_k \mid > r + \varepsilon\}$ for $\varepsilon > 0$. Then $\delta(E) = 0$ and we have

$$\mid \sum_k t(n,k,p)(z - x_k) \mid \leq \sup_k \mid z - x_k \mid \sum_{k \in E} \mid t(n,k,p) \mid + \ (r + \varepsilon) \sum_{k \notin E} \mid t(n,k,p) \mid . \tag{4.3.1.5}$$

Therefore by conditions (4.3.1.2) and (4.3.1.3), we get

$$\limsup_p \sup_n \mid \sum_k t(n,k,p)(z-x_k) \mid \leq r+\varepsilon.$$

Hence by (4.3.1.4) we have

$$\mid w-z \mid \leq r+\varepsilon$$

and since ε is arbitrary

$$\mid w-z \mid \leq r = st-\limsup_k \mid z-x_k \mid,$$

i.e. $w \in S_x^\star(z)$. Hence $w \in st-\text{core}\{x\}$, so that

$$B-\text{core}\{Ax\} \subseteq st-\text{core}\{x\}.$$

This completes the proof of the theorem.

Our next result is an analogue of Theorem 3.2.3 as well as Theorem 3.2.4, which is a slight generalization of the previous result.

Theorem 4.3.2. Let $T=(t_{jk})$ be a normal matrix. Let $A=(a_{nj})$ be any matrix. In order that whenever Tx is bounded Ax should exist and be bounded and satisfy

(4.3.2.1) $$B-\text{core}\{Ax\} \subseteq st-\text{core}\{Tx\}$$

it is necessary and sufficient that

(4.3.2.2) $(c_{nk})=C=AT^{-1}$ exists;

(4.3.2.3) C is almost regular, and

$$\lim_p \sum_{k\in E} \mid b(n,k,p) \mid = 0 \text{ uniformly in } n, \text{ whenever } \delta(E)=0 \text{ for } E \subseteq \mathbb{N};$$

(4.3.2.4) $\limsup_p \sup_n \sum_k \mid b(n,k,p) \mid = 1$, where

$$b(n,k,p) = \frac{1}{p+1}\sum_{i=0}^{p} c_{n+i,k} \ ;$$

(4.3.2.5) for any fixed n,

$$\sum_{k=0}^{m} \left| \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} \right| \to 0 \text{ as } m \to \infty .$$

Proof . Necessity. Let (4.3.2.1) hold and $A_n(x)$ exist for every n whenever $Tx \in \ell_\infty$. Then by Lemma 2.1.5 it follows that conditions (4.3.2.2) and (4.3.2.5) hold. Further by the same Lemma, we obtain $Ax = Cy$, where $y = Tx$. Since $Ax \in \ell_\infty$, we have $Cy \in \ell_\infty$. Therefore (4.3.2.1) implies that

$$B - \text{core}\{Cy\} \subseteq st - \text{core}\{y\}.$$

Hence using Theorem 4.3.1, we see that conditions (4.3.2.2) and (4.3.2.4) hold.

Sufficiency. Let the conditions (4.3.2.2)-(4.3.2.5) hold. Then obviously the conditions of Lemma 2.1.5 are satisfied and so $Cy \in \ell_\infty$, hence $Ax \in \ell_\infty$. Now using Theorem 4.3.1, we obtain

$$B - \text{core}\{Cy\} \subseteq st - \text{core}\{y\}$$

and consequently

$$B - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\},$$

since $y = Tx$ and $Cy = Ax$.

This completes the proof of the theorem

We can also easily prove the following results analogous to Theorems 4.2.1 and 4.2.2.

Theorem 4.3.3. Let A and T be same as in Theorem 4.3.2. Then

(4.3.3.1) $$B - \text{core}\{Ax\} \subseteq K - \text{core}\{Tx\}$$

if and only if (4.3.2.2) and (4.3.2.5) hold

(4.3.3.2) C is almost regular;

(4.3.3.3) $\limsup_p \sup_n \sum_k \mid \frac{1}{p+1}\sum_{i=0}^{p} c_{n+i,k} \mid = 1.$

Theorem 4.3.4. Let A and T be same as in Theorem 4.3.2. Then

(4.3.4.1) $$B - \text{core}\{Ax\} \subseteq B - \text{core}\{Tx\}$$

if and only if (4.3.2.2), (4.3.2.5) and (4.3.3.3) hold

(4.3.4.2) C is F-regular .

Remark 4.3.5. If we take $T = I$, the unit matrix, then Theorem 4.3.2 reduces to Theorem 4.3.1 and Theorem 4.3.3 and 4.3.4 reduce to that of Orhan [44].

Theorem 4.3.6. If $\|A\| < \infty$, then for every $x \in \ell_\infty$

(4.3.6.1) $$st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\}$$

if and only if

(4.3.6.2) $A \in (f, st \cap \ell_\infty)_{reg}$;

(4.3.6.3) $st - \lim_n \sum_{k\in E} a_{nk} = 1$, whenever $\mathbb{N}\backslash E$ is finite for $E \subseteq \mathbb{N}$.

Proof . Necessity. Let (4.3.6.1) hold and x be almost convergent to L. Then

$$\{L\} = B - \text{core}\{x\} \supseteq st - \text{core}\{Ax\}.$$

Since $\|A\| < \infty$ implies $Ax \in \ell_\infty$ for $x \in \ell_\infty$, we have

$$-q(-x) \leq st - \liminf Ax \leq st - \limsup Ax \leq q(x).$$

But $-q(-x) = q(x) = L$, so that

$$st - \lim Ax = f - \lim x = L.$$

Hence $A \in (f, st \cap \ell_\infty)_{reg}$, i.e. condition (4.3.6.2) holds.

To prove (4.3.6.3), let us define $x = (x_k) \in \ell_\infty$ by

$$x_k = \begin{cases} 1 & , \text{ if } k \in E, \\ 0 & , \text{ otherwise,} \end{cases}$$

where $E \subseteq \mathbb{N}$ such that $\mathbb{N} \backslash E$ is finite. Then

$$B - \text{core}\{x\} = \{1\}.$$

Since $Ax \in \ell_\infty$, Ax has at least a statistical cluster point. Therefore by proposition 1.4.2, $st - \text{core}\{Ax\} \neq \emptyset$. Since

$$st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\} = \{1\},$$

we have $st - \text{core}\{Ax\} = \{1\}$ and 1 is the only statistical cluster point of Ax. Hence

$$st - \lim Ax = 1,$$

i.e.

$$st - \lim_n \sum_{k \in E} a_{nk} = 1, \text{ whenever } \mathbb{N} \backslash E \text{ is finite.}$$

Therefore (4.3.6.3) holds.

Sufficiency. Suppose that the conditions (4.3.6.2) and (4.3.6.3) hold and $w \in st - \text{core}\{Ax\}$. Then for any $z \in \mathbb{C}$ we have

$$\begin{aligned} | w - z | &\leq st - \limsup_n | z - A_n(x) | \\ &= st - \limsup_n | z - \sum_k a_{nk} x_k | \\ &\leq st - \limsup_n | \sum_k a_{nk}(z - x_k) | \\ &\quad + \ st - \limsup_n | z | \, | 1 - \sum_k a_{nk} | \\ &= st - \limsup_n | \sum_k a_{nk}(z - x_k) | \ , \quad \text{by (4.3.6.3).} \end{aligned}$$

Therefore for an index set $N = \{n_i\}$ such that $\delta(N) = 1$,

$$(4.3.6.4) \qquad |\, w - z \,| \leq \limsup_i |\, \sum_k a_{n_i k}(z - x_k) \,| \,.$$

Now proceeding as in the proof of sufficiency part of Theorem 2.3.4, we obtain

$$(4.3.6.5) \qquad \limsup_i \sum_k a_{n_i k}(z - x_k) = \limsup_i \sum_k a_{n_i k}(z - t_{pn}(x)).$$

Now, let $r = \limsup_p \sup_n |\, t_{pn}(x) - z \,|$ and $E = \{k : |\, t_{pn}(x) - z \,| > r + \varepsilon\}$ for $\varepsilon > 0$. Then $\delta(E) = 0$ as E is finite. Therefore

$$|\, \sum_k a_{n_i k}(z - x_k) \,| = |\, \sum_k a_{n_i k}(z - t_{pn}(x)) \,|$$

$$\leq \sup_n |\, z - t_{pn}(x) \,| \,|\, \sum_{k \in E} a_{n_i k} \,| + (r + \varepsilon) \,|\, \sum_{k \notin E} a_{n_i k} \,| \,.$$

From (4.3.6.2) and (4.3.6.3), we get

$$\limsup_i |\, \sum_k a_{n_i k}(z - x_k) \,| \leq r + \varepsilon$$

and so by (4.3.6.4) we have

$$|\, w - z \,| \leq r + \varepsilon.$$

Since ε is arbitrary,

$$|\, w - z \,| \leq r = \limsup_p \sup_n |\, t_{pn}(x) - z \,|,$$

i.e. $w \in B_x^*(z)$. Hence $w \in B - \text{core}\{x\}$, so that

$$st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\}.$$

Remark 4.3.7. In Example 4.4.3, we shall see that the condition (4.3.6.3) can not be replaced by

$$(4.3.7.1) \qquad st - \lim_n \sum_{k \in E} a_{nk} = 1 \text{ for any set } E \subseteq \mathbb{N} \text{ such that } \delta(E) = 1.$$

Our next theorem is an analogue of Theorem 3.3.9.

Theorem 4.3.8. Let $T = (t_{jk})$ be a normal matrix . Let $A = (a_{nj})$ be any matrix. In order that whenever Tx is bounded Ax should exist and be bounded and satisfy

(4.3.8.1) $$st - \text{core}\{Ax\} \subseteq B - \text{core}\{Tx\}$$

it is necessary and sufficient that

(4.3.8.2) $(c_{nk}) = C = AT^{-1}$ exists;

(4.3.8.3) $C \in (f, st \cap \ell_\infty)_{reg}$;

(4.3.8.4) $st - \lim_n \sum_{k \in E} c_{nk} = 1$ whenever $\mathbb{N} \backslash E$ is finite;

(4.3.8.5) for any fixed n,

$$\sum_{k=0}^{m} \mid \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} \mid \to 0 \text{ as } m \to \infty.$$

Remark 4.3.9. If $T = I$, the unit matrix, then Theorem 4.3.8 is reduced to Theorem 4.3.6.

4.4. Examples

Example 4.4.1. In support of our main result Theorem 4.3.1, we will show that the condition

$$\lim_p \sum_{k \in E} \mid t(n,k,p) \mid = 0 \quad \text{uniformly in } n,$$

of (4.3.1.2) can not be changed whenever $\delta(E) = 0$.

Let $A = (a_{nk})$ be defined by

$$a_{nk} = \begin{cases} 2 & , \text{ if } n \text{ is even and } k = n^2, \\ 0 & , \text{ otherwise.} \end{cases}$$

Then

$$\sum_k a_{nk} = \begin{cases} 2 & \text{, if } n \text{ is even,} \\ 0 & \text{, otherwise.} \end{cases}$$

We see that $A \in (c, f)_{reg}$. Now, let $E = \{k = n^2 : k \in \mathbb{N}\}$. Then $\delta(E) = 0$, and

$$\sum_{k \in E} a_{nk} = \begin{cases} 2 & \text{, if } n \text{ is even,} \\ 0 & \text{, otherwise.} \end{cases}$$

Therefore

$$\lim_p \sum_{k \in E} \mid t(n, k, p) \mid = 1 \text{ uniformly in } n,$$

and

$$\lim_p \sup_n \sup \sum_k \mid t(n, k, p) \mid = 1.$$

Let $x = (x_k)$ be defined by

$$x_k = \begin{cases} 3 & \text{, if } k \text{ is not square,} \\ 1 & \text{, if } k \text{ is square.} \end{cases}$$

Then

$$st - \text{core}\{x\} = \{3\}$$

and

$$\sum_{k \in E} t(n, k, p) x_k = \begin{cases} 2 & \text{, if } n \text{ is even,} \\ 0 & \text{, otherwise.} \end{cases}$$

Therefore

$$B - \text{core}\{Ax\} = \{1\}.$$

Hence

$$B - \text{core}\{Ax\} \not\subset st - \text{core}\{x\}.$$

Example 4.4.2. Let $A = (a_{nk})$ be defined by

$$a_{nk} = \begin{cases} \frac{2}{n} & , \text{ if } n \text{ is even and } k \leq n, \\ \\ 0 & , \text{ otherwise.} \end{cases}$$

Then

$$\sum_k a_{nk} = \begin{cases} 2 & , \text{ if } n \text{ is even,} \\ 0 & , \text{ otherwise.} \end{cases}$$

We see that $A \in (st \cap \ell_\infty, f)_{reg}$. Hence $A \in (c, f)_{reg}$. Moreover, for any set $E \subseteq \mathbb{N}$ such that $\delta(E) = 0$,

$$\lim_p \sum_{k \in E} \mid t(n,k,p) \mid = 0 \;\; \text{uniformly in } n,$$

and

$$\lim_p \sup_n \sup \sum_k \mid t(n,k,p) \mid = 1.$$

Hence for any bounded sequence, e.g. $x = (x_k) = (1, 0, 1, 0, \cdots)$, we have

$$\sum_k t(n,k,p) x_k = \begin{cases} 1 & , \text{ if } n \text{ is even,} \\ 0 & , \text{ otherwise} \end{cases}$$

and so

$$B - \text{core}\{Ax\} = \{\frac{1}{2}\}.$$

Therefore we have

$$\{\frac{1}{2}\} = B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\} = [0, 1].$$

Example 4.4.3. Let $A = (a_{nk})$ be defined by

$$a_{nk} = \begin{cases} \frac{1}{2} & , \text{ if } n \text{ is a nonsquare and } k = n^2 \text{ or } n^2+1, \\ \\ 0 & , \text{ otherwise.} \end{cases}$$

Then

$$\sum_k a_{nk} = \begin{cases} 1 & , \text{ if } n \text{ is a nonsquare,} \\ 0 & , \text{ otherwise,} \end{cases}$$

and $A \in (f, st \cap \ell_\infty)_{reg}$. Further, for any set $E \subseteq \mathbb{N}$ such that $\mathbb{N} \backslash E$ is finite, we have

$$st - \lim_n \sum_{k \in E} a_{nk} = 1.$$

Then for any bounded sequence x we have

$$st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\}.$$

Now, let $E = \{k \neq n^2 \text{ and } k \neq n^2 + 1 : k \in \mathbb{N}\}$. Then $\delta(E) = 1$ and we have

$$\sum_{k \in E} a_{nk} = 0, \quad \text{for all } n.$$

Hence

$$st - \lim_n \sum_{k \in E} a_{nk} = 0.$$

Further, for any bounded sequence, say, $x = (1, 0, 1, 0, \cdots)$, we have $B - \text{core}\{x\} = \{\frac{1}{2}\}$ and

$$\sum_k a_{nk}x_k = \begin{cases} \frac{1}{2} & , \text{ if } n \text{ is a nonsquare,} \\ \\ 0 & , \text{ otherwise.} \end{cases}$$

Therefore, $st - \lim Ax = \frac{1}{2}$, i.e.

$$st - \text{core}\{Ax\} = \{\frac{1}{2}\} = B - \text{core}\{x\},$$

but (4.3.7.1) does not hold.

CHAPTER V

GENERALIZED STATISTICAL CONVERGENCE

5.1. Introduction

Let $A = (a_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix. Freedman and Sember [17] defined A-density of the set $K \subseteq \mathbb{N}$ if

$$\delta_A(K) = \lim_n \sum_{k \in K} a_{nk} \quad \text{exists.}$$

A sequence $x = (x_k)$ is said to be A-statistically convergent to ℓ if for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in K : | x_k - \ell | \geq \varepsilon\}$ has A-density zero (cf. Buck [3], Connor [6,7], Kolk [30]).

Kolk [32] generalized the idea of A-statistical convergence to $\mathcal{B}$-statistical convergence by using the idea of $\mathcal{B}$-summability (or $F_{\mathcal{B}}$-convergence) due to Steiglitz [53].

Let $\mathcal{B} = (B_i)$ be a sequence of infinite matrices with $B_i = (b_{nk}(i))$. Then $x \in \ell_\infty$ is said to be $F_{\mathcal{B}}$-convergent (or $\mathcal{B}$-summable) to the value $\mathcal{B} - \lim x$ (denotes the generalized limit) if

$$\lim_n (B_i x)_n = \lim_n \sum_k b_{nk}(i) = \mathcal{B} - \lim x, \text{ uniformly in } i \geq 0.$$

The method $\mathcal{B}$ is regular (cf. Steiglitz [53], Bell [2]) if and only if

(i) $\|\mathcal{B}\| < \infty$;

(ii) $\lim_n b_{nk}(i) = 0$ for all $k \geq 1$. uniformly in i;

(iii) $\lim_n \sum_k b_{nk}(i) = 1$, uniformly in i,

where

$$\|\mathcal{B}\| = \sup_{n,i} \sum_k \mid b_{nk}(i) \mid < \infty$$

to mean that, there exists a constant M such that

$$\sum_k \mid b_{nk}(i) \mid \leq M \text{ for all } n, i$$

and the series $\sum_k \mid b_{nk}(i) \mid$ converges uniformly in i for each n.

Kolk [32] introduced the following:

An index set K is said to have $\mathcal{B}$-density $\delta_{\mathcal{B}}(K)$ equal to d, if the characteristic sequence of K is $\mathcal{B}$-summable to d, i.e.

$$\lim_n \sum_{k \in K} b_{nk}(i) = d, \text{ uniformly in } i.$$

Let $\Re^+$ denote the set of all regular methods $\mathcal{B}$ with $b_{nk}(i) \geq 0$ for all n, k and i.

Let $\mathcal{B} \in \Re^+$. A sequence $x = (x_k)$ is called $\mathcal{B}$-statistically convergent to the number ℓ, if for every $\varepsilon > 0$

$$\delta_{\mathcal{B}} \mid \{k : \mid x_k - \ell \mid \geq \varepsilon\} \mid = 0$$

and we write $st_{\mathcal{B}} - \lim x = \ell$. We denote by $st(\mathcal{B})$ the space of all $\mathcal{B}$-statistically convergent sequences.

In particular, if $\mathcal{B} = (C_1)$, the Cesàro matrix, then $\mathcal{B}$-statistical convergence is reduced to the usual statistical convergence. For $\mathcal{B} = (A)$, it is reduced to A-statistical convergence. For $\mathcal{B} = (A_\theta)$, it is reduced to lacunary statistical convergence (cf. Fridy and Orhan [21]), where $A_\theta = (a^\theta_{ri})$ with

$$a^\theta_{ri} = \begin{cases} \frac{1}{h_r} & , \text{ if } i \in I_r, \\ 0 & , \text{ otherwise}; \end{cases}$$

$\theta = \{k_r\}_{r=0}^{\infty}$ is an increasing sequence of integers such that $k_o = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $I_r = (k_{r-1}, k_r]$.

For $\mathcal{B} = \mathcal{B}_1$, it is reduced to uniform statistical convergence (cf. Pehlivan [46]), where $\mathcal{B}_1 = (b^1_{nk}(i))$ with

$$b^1_{nk}(i) = \begin{cases} \frac{1}{n} & , \text{ if } 1+i \leq k \leq n+i, \\ \\ 0 & , \text{ otherwise.} \end{cases}$$

5.2. $\mathcal{B}$-Statistical Cluster and $\mathcal{B}$-Statistical Limit Points

We start this section with the following example to show that neither of the two methods, statistical convergence and $\mathcal{B}$-statistical convergence, implies the other.

Example 5.2.1. Let us consider the sequence of infinite matrices $\mathcal{B} = (B_i)$ with

$$b_{nk}(i) = \begin{cases} \frac{1}{i} + \frac{1}{in} & , \text{ if } k = n^2, \\ 1 - \frac{n}{i(n+1)} & , \text{ if } k = n^2 + 1, \\ 0 & , \text{ otherwise.} \end{cases}$$

It is not difficult to see that $\mathcal{B} \in \Re^+$.

Now let us define the sequences $x = (x_k)$ and $y = (y_k)$ by

$$x_k = \begin{cases} 0 & , \text{ if } k = n^2, \\ \frac{1}{k} & , \text{ if } k = n^2 + 1, \\ k & , \text{ otherwise,} \end{cases}$$

and

$$y_k = \begin{cases} k & , \text{ if } k = n^2, \\ 0 & , \text{ if } k = n^2 + 1, \\ 1 & , \text{ otherwise.} \end{cases}$$

Then x is not statistically convergent to zero as $\delta\{k : \mid x_k \mid \geq \varepsilon\} \neq 0$ but it is $\mathcal{B}$-statistically convergent to zero; and on the other hand y is statistically convergent but not $\mathcal{B}$-statistically convergent.

We give some analogue definitions for the method $\mathcal{B}$.

Definition 5.2.2. Let $\mathcal{B} \in \Re^+$. The number γ is said to be $\mathcal{B}$-statistical cluster point of a sequence x if for every $\varepsilon > 0$ the set $\{k : \mid x_k - \gamma \mid < \varepsilon\}$ does not have $\mathcal{B}$-density zero.

Definition 5.2.3. Let $\mathcal{B} \in \Re^+$. The number λ is said to be $\mathcal{B}$-statistical limit point of a sequence x if there is a subsequence of x which converges to λ such that whose indices do not have $\mathcal{B}$-density zero.

We denote by $\Gamma_x(\mathcal{B})$ the set of $\mathcal{B}$-statistical cluster points and by $\Lambda_x(\mathcal{B})$ the set of $\mathcal{B}$-statistical limit points of x.

From the above examples we can see that $\Gamma_x(\mathcal{B}) = \{0\}$ and $\Lambda_x(\mathcal{B}) = \{0\}, \Gamma_y(\mathcal{B}) = \{0\}$ and $\Lambda_y(\mathcal{B}) = \{0\}$.

Note that for $\mathcal{B} = (A)$ in Definitions 5.2.2 and 5.2.3, we get A-statistical cluster point and A-statistical limit point (cf. Connor and Kline [8]). For $\mathcal{B} = (C_1)$ these are reduced to the usual statistical cluster point and statistical limit point respectively (cf. Fridy [19]).

Throughout this chapter we will consider $\mathcal{B} \in \Re^+$.

Definition 5.2.4. Let us write

$$G_x = \{g \in \mathbb{R} : \delta_{\mathcal{B}}\{k : x_k > g\} \neq 0\},$$

and

$$F_x = \{f \in \mathbb{R} : \delta_{\mathcal{B}}\{k : x_k < f\} \neq 0\},$$

for a number sequence $x = (x_k)$. Then we define the $\mathcal{B}$-statistical limit superior and $\mathcal{B}$-statistical limit inferior of x as follows:

$$st_{\mathcal{B}} - \limsup x = \begin{cases} \sup G_x & , \text{ if } G_x \neq \emptyset, \\ \\ -\infty & , \text{ if } G_x = \emptyset, \end{cases}$$

and

$$st_{\mathcal{B}} - \liminf x = \begin{cases} \inf F_x & , \text{ if } F_x \neq \emptyset, \\ \\ +\infty & , \text{ if } F_x = \emptyset. \end{cases}$$

Definition 5.2.5. The number sequence x is said to be $\mathcal{B}-$ statistically bounded if there is a number M such that

$$\delta_{\mathcal{B}}\{k : \mid x_k \mid > M\} = 0.$$

Note that for $\mathcal{B} = (A)$ in Definitions 5.2.4 and 5.2.5, we get A-statistical limit superior. A-statistical limit inferior and A-statistically bounded (cf. Demirci [11]). For $\mathcal{B} = (C_1)$. these are reduced to the usual statistical limit superior, statistical limit inferior and statistically bounded respectively (cf. Fridy [22]).

Example 5.2.6. Let us consider the same $\mathcal{B}$ as defined in Example 5.2.1. Define the sequence $z = (z_k)$ by

$$z_k = \begin{cases} 0 & , \text{ if } k = n^2, \\ 1 & , \text{ if } k = n^2 + 1, \\ k & , \text{ otherwise.} \end{cases}$$

Here we see that z is unbounded above but it is $\mathcal{B}$-statistically bounded, since $\delta_{\mathcal{B}}\{k : \mid z_k \mid > 1\} = 0$. Also z is not statistically bounded. Thus

$G_z = (-\infty, 1)$ and $F_z = (0, \infty)$ so that $st_{\mathcal{B}} - \limsup z = 1$ and $st_{\mathcal{B}} - \liminf z = 0$. Moreover $\Gamma_z(\mathcal{B}) = \{0, 1\} = \Lambda_z(\mathcal{B})$ and z is neither $\mathcal{B}$-statistically nor statistically convergent.

In this example we see that z is $\mathcal{B}$-statistically bounded but not $\mathcal{B}$-statistically convergent. On the other hand in Example 5.2.1, y is statistically convergent but not $\mathcal{B}$-statistically bounded.

Also note that $st_{\mathcal{B}} - \limsup z$ equals the greatest element of $\Gamma_z(\mathcal{B})$ while $st_{\mathcal{B}} - \liminf z$ is the least element of $\Gamma_z(\mathcal{B})$. This observation suggest the following result which can be proved by straightforward least upper bound argument.

Theorem 5.2.7. (a) If $s_1 = st_{\mathcal{B}} - \limsup x$ is finite, then for every positive number ε

(5.2.7.1) $\delta_{\mathcal{B}}\{k : x_k > s_1 - \varepsilon\} \neq 0$ and $\delta_{\mathcal{B}}\{k : x_k > s_1 + \varepsilon\} = 0$.

Conversely, if (5.2.7.1) holds for every $\varepsilon > 0$ then $s_1 = st_{\mathcal{B}} - \limsup x$.

(b) If $s_2 = st_{\mathcal{B}} - \liminf x$ is finite, then for every positive number ε

(5.2.7.2) $\delta_{\mathcal{B}}\{k : x_k < s_2 + \varepsilon\} \neq 0$ and $\delta_{\mathcal{B}}\{k : x_k < s_2 - \varepsilon\} = 0$.

Conversely, if (5.2.7.2) holds for every $\varepsilon > 0$ then $s_2 = st_{\mathcal{B}} - \liminf x$.

From the Definition 5.2.2 we see that the above theorem can be interpreted as saying that $st_{\mathcal{B}} - \limsup x$ and $st_{\mathcal{B}} - \liminf x$ are the greatest and least $\mathcal{B}$-statistical cluster points of x.

Note that $\mathcal{B}$-statistical boundedness implies that $st_{\mathcal{B}} - \limsup x$ and $st_{\mathcal{B}} - \liminf x$ are finite, so that properties (5.2.7.1) and (5.2.7.2) of Theorem 5.2.7 hold good.

In the following section, we produce $\mathcal{B}$-analogues of the results of Fridy and Orhan [22].

5.3. Main Theorems

Throughout the chapter by $\delta_{\mathcal{B}}(K) \neq 0$ we mean that either $\delta_{\mathcal{B}}(K) > 0$ or K fails to have $\mathcal{B}$-density.

Theorem 5.3.1. For any real number sequence x

$$st_{\mathcal{B}} - \liminf x \leq st_{\mathcal{B}} - \limsup x.$$

Proof. First consider the case in which $st_{\mathcal{B}} - \limsup x = -\infty$. This implies that $G_x = \emptyset$. Therefore for every $g \in \mathbb{R}$, $\delta_{\mathcal{B}}\{k : x_k > g\} = 0$, which implies that $\delta_{\mathcal{B}}\{k : x_k \leq g\} = 1$. So that for every $f \in \mathbb{R}$, $\delta_{\mathcal{B}}\{k : x_k < f\} \neq 0$. Hence $st_{\mathcal{B}} - \liminf x = -\infty$.

Now consider $st_{\mathcal{B}} - \limsup x = +\infty$. This implies that for every $g \in \mathbb{R}$, $\delta_{\mathcal{B}}\{k : x_k > g\} \neq 0$. This means that $\delta_{\mathcal{B}}\{k : x_k \leq g\} = 0$. Therefore for every $f \in \mathbb{R}$, $\delta_{\mathcal{B}}\{k : x_k < f\} = 0$, which implies that $F_x = \emptyset$. Hence $st_{\mathcal{B}} - \liminf x = +\infty$.

Next we assume that $s_1 = st_{\mathcal{B}} - \limsup x < \infty$ and let $s_2 = st_{\mathcal{B}} - \liminf x$. Given $\varepsilon > 0$ we show that $s_1 + \varepsilon \in F_x$, so that $s_2 \leq s_1 + \varepsilon$. By Theorem 5.2.7(a), $\delta_{\mathcal{B}}\{k : x_k > s_1 + \varepsilon/2\} = 0$, since $s_1 = lub\ G_x$. This implies that $\delta_{\mathcal{B}}\{k : x_k \leq s_1 + \epsilon/2\} = 1$, which in turn gives $\delta_{\mathcal{B}}\{k : x_k < s_1 + \varepsilon\} = 1$. Hence $s_1 + \varepsilon \in F_x$ and so that $s_2 \leq s_1 + \varepsilon$, i.e. $s_2 \leq s_1$ since ε was arbitrary.

Remark 5.3.2. (i) For any number sequence x,

$$\liminf x \leq st_{\mathcal{B}} - \liminf x \leq st_{\mathcal{B}} - \limsup x \leq \limsup x.$$

(ii) One can not say that $st_{\mathcal{B}} - \limsup x$ is equal to the greatest $\mathcal{B}$-statistical limit point of x. Consider the Cesàro matrix of order $r > -1$

$$C_{nk}^r = \begin{cases} A_{n-k}^{r-1}/A_n^r & , \ 0 \le k \le n, \\ \\ 0 & , \ k > n, \end{cases}$$

where $A_n^r = (r+1)(r+2)\cdots(r+n)/n!$ for $n \geq 1$ and $A_0^r = 1$.

Let x be the sequence $\{0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \cdots\}$ (cf. Example 4 of Fridy [19]). Then we have

$$\Gamma_x(\mathcal{B}) = [0, 1] \text{ and } \Lambda_x(\mathcal{B}) = \emptyset,$$

$$st_{\mathcal{B}} - \limsup x = 1 \text{ as } \delta_{\mathcal{B}}\{k : x_k > 1 - \varepsilon\} \neq 0.$$

Theorem 5.3.3. For any number sequence x,

$$\mathcal{B}\text{-statistical boundedness} \implies \mathcal{B}\text{-statistical convergence}$$

if and only if

$$st_{\mathcal{B}} - \liminf x = st_{\mathcal{B}} - \limsup x.$$

Proof. Let $s_1 = st_{\mathcal{B}} - \limsup x$ and $s_2 = st_{\mathcal{B}} - \liminf x$. First assume that $st_{\mathcal{B}} - \lim x = \ell$ and $\varepsilon > 0$. Then $\delta_{\mathcal{B}}\{k : \mid x_k - \ell \mid \geq \varepsilon\} = 0$, so that $\delta_{\mathcal{B}}\{k : x_k > \ell + \varepsilon\} = 0$, which implies that $s_1 \leq \ell$. Also $\delta_{\mathcal{B}}\{k : x_k < \ell - \varepsilon\} = 0$, which implies that $\ell \leq s_2$. By Theorem 5.3.1, we finally have $s_1 = s_2$.

Conversely, suppose that $s_1 = s_2 = \ell$ and x be $\mathcal{B}$-statistically bounded. Then for $\varepsilon > 0$, by Theorem 5.2.7, we have $\delta_{\mathcal{B}}\{k : x_k > \ell + \varepsilon/2\} = 0$, and $\delta_{\mathcal{B}}\{k : x_k < \ell - \varepsilon/2\} = 0$. Hence $st_{\mathcal{B}} - \lim x = \ell$.

Theorem 5.3.4. If the number sequence x is bounded above and $\mathcal{B}$-summable to the number $\ell = st_{\mathcal{B}} - \limsup x$, then x is $\mathcal{B}$-statistically convergent to ℓ.

Proof. Suppose that x is not $\mathcal{B}$-statistically convergent to ℓ. Then by Theorem 5.3.3, $st_{\mathcal{B}} - \liminf x < \ell$, so there is a number $M < \ell$ such that

$\delta_{\mathcal{B}}\{k : x_k < M\} \neq 0$. Let $K' = \{k : x_k < M\}$. Then for every $\varepsilon > 0$, $\delta_{\mathcal{B}}\{k : x_k > \ell + \varepsilon\} = 0$.

Write $K'' = \{k : M \leq x_k \leq \ell + \varepsilon\}$ and $K''' = \{k : x_k > \ell + \varepsilon\}$, and let $G = \sup_k x_k < \infty$. Since $\delta_{\mathcal{B}}(K') \neq 0$, there are many n such that

$$\limsup_{n} \sum_{k \in K'} b_{nk}(i) \geq d > 0,$$

and for each n, i

$$\sum_{k=1}^{\infty} | \, b_{nk}(i) x_k \, | < \infty.$$

Now

$$\sum_{k=1}^{\infty} b_{nk}(i) x_k = \Big(\sum_{k \in K'} + \sum_{k \in K''} + \sum_{k \in K'''} \Big) b_{nk}(i) x_k$$

$$\leq M \sum_{k \in K'} b_{nk}(i) + (\ell + \varepsilon) \sum_{k \in K''} b_{nk}(i) + G \sum_{k \in K'''} b_{nk}(i)$$

$$= M \sum_{k \in K'} b_{nk}(i) + (\ell + \varepsilon) \sum_{k=1}^{\infty} b_{nk}(i) - (\ell + \varepsilon) \sum_{k \in K'} b_{nk}(i) + O(1)$$

$$= - \sum_{k \in K'} b_{nk}(i) \big[-M + (\ell + \varepsilon) \big] + (\ell + \varepsilon) \sum_{k=1}^{\infty} b_{nk}(i) + O(1)$$

$$\leq \ell \sum_{k=1}^{\infty} b_{nk}(i) - d(\ell - M) + \varepsilon \Big(\sum_{k=1}^{\infty} b_{nk}(i) - d \Big) + O(1).$$

Since ε is arbitrary, it follows that

$$\liminf \mathcal{B}x \leq \ell - d(\ell - M) < \ell.$$

Hence x is not $\mathcal{B}$-summable to ℓ.

This completes the proof of the theorem.

The following is the dual statement of Theorem 5.3.4.

Theorem 5.3.5. If the number sequence x is bounded below and $\mathcal{B}$-summable to the number $\ell = st_{\mathcal{B}} - \liminf x$, then x is $\mathcal{B}$-statistically convergent to ℓ.

Note: Like Fridy and Orhan [22], we can see that in the above Theorems 5.3.4 and 5.3.5, the boundedness of x can not be omitted or even replaced by the $\mathcal{B}$-statistical boundedness. For consider the matrix $\mathcal{B} = (C^{r}_{nk})$ of Remark 5.3.2(ii) and the sequence x of Example 2 [22].

CHAPTER VI

ALMOST CONVERGENCE AND A CORE THEOREM FOR DOUBLE SEQUENCES

6.1. Introduction

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [47]. A double sequence $x = (x_{jk})_{j,k=0}^{\infty}$ is said to be convergent in the Pringsheim's sense or P-convergent if for every $\varepsilon > 0$ there exists $N \in I\!N$ such that $| x_{jk} - \ell | < \varepsilon$ whenever $j, k > N$ and we denote by $P - \lim x = \ell$. The number ℓ is called the Pringsheim limit of x.

More exactly we say that a double sequence (x_{jk}) converges to a finite number ℓ if x_{jk} tends to ℓ as both j and k tend to ∞ independently of one another.

We denote the space of P-convergent sequences by c_2.

A double sequence $x = (x_{jk})$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in I\!N$ such that $| x_{pq} - x_{jk} | < \varepsilon$ for all $p \geq j \geq I\!N$ and $q \geq k \geq N$.

A double sequence x is bounded if there exists a positive number M such that $| x_{jk} | < M$ for all j and k, i.e. if

$$\|x\|_{(\infty,2)} = \sup_{j,k} | x_{jk} | < \infty .$$

We denote the set of all bounded double sequences by ℓ_{∞}^{2}.

Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded but every convergent real (or complex) double sequence is Cauchy.

Let $A = (a_{jk}^{mn})_{j,k=0}^{\infty}$ be a doubly infinite matrix of real numbers for all $m, n = 0, 1, \cdots$. Forming the sums

$$y_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} \, x_{jk} \ ,$$

called the A-means of the double sequence x, yields a method of summability.

We say that a sequence x is A-summable to the limit ℓ if the A-means exist for all $m, n = 0, 1, \cdots$ in the sense of Pringsheim's convergence :

$$\lim_{p,q\to\infty} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn} \; x_{jk} = y_{mn} \; ,$$

and

$$\lim_{m,n\to\infty} y_{mn} = \ell \; .$$

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. In 1926 Robinson [48] presented a four dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness :

A four dimensional matrix A is said to be bounded-regular or RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The following is a four dimensional analogue of the well-known Silverman-Töeplitz theorem:

Theorem 6.1.1 (Hamilton [25], Robinson [48]). The four dimensional matrix A is bounded-regular or RH-regular if and only if

$RH_1: \quad P - \lim_{m,n} \; a_{jk}^{mn} = 0 \quad (j, k = 0, 1, \cdots);$

$RH_2: \quad P - \lim_{m,n} \sum_{j,k=0,0}^{\infty,\infty} \; a_{jk}^{mn} = 1;$

$RH_3: \quad P - \lim_{m,n} \sum_{j=0}^{\infty} \mid a_{jk}^{mn} \mid = 0 \quad (k = 0, 1, \cdots);$

$RH_4: \quad P - \lim_{m,n} \sum_{k=0}^{\infty} \mid a_{jk}^{mn} \mid = 0 \quad (j = 0, 1, \cdots);$

$RH_5: \quad \sum_{j,k=0,0}^{\infty,\infty} \mid a_{jk}^{mn} \mid \leq C < \infty \quad (m, n = 0, 1, \cdots)$, where C is constant.

Note that RH_1 is a consequence of each of RH_3 and RH_4.

Recently in [45], Patterson extended the idea of Knopp's core theorem for double sequences by defining the Pringsheim core as follows :

Let $P - C_n\{x\}$ be the least closed convex set that includes all points x_{jk} for $j, k > n$; then the Pringsheim core of the double sequence $x = (x_{jk})$ is the set

$$P - C\{x\} = \bigcap_{n=1}^{\infty} [P - C_n\{x\}] \ .$$

Note that the Pringsheim core of a real-valued bounded double sequence is the closed interval $[P - \liminf x, P - \limsup x]$.

In this regard, Patterson [45] proved the following :

Theorem 6.1.2. If A is a four dimensional matrix, then for all real-valued double sequences x ,

(6.1.2.1) $$P - \limsup Ax \leq P - \limsup x$$

if and only if

(6.1.2.2) A is RH-regular;

(6.1.2.3) $P - \lim_{mn} \sum_{j,k=0,0}^{\infty,\infty} \mid a_{jk}^{mn} \mid = 1$.

In the present chapter we define the MR-core of a double sequence by using the idea of almost convergence introduced and studied by Moricz and Rhoades [40], and then proved an analogue of Theorem 6.1.2.

6.2. Almost Convergence and MR-Core

The notion of almost convergence for single sequences was introduced by Lorentz [35]. Recently Moricz and Rhoades [40] extended this idea for double sequences.

A double sequence $x = (x_{jk})_{j,k=0}^{\infty}$ of real numbers is said to be almost convergent to a limit L if

$$\lim_{p,q\to\infty} \sup_{m,n\geq 0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - L \right| = 0 ,$$

that is, the average value of (x_{jk}) taken over any rectangle $\{(j,k) : m \leq j \leq m+p-1;\ n \leq k \leq n+q-1\}$ tends to L as both p and q tend to ∞, and this convergence is uniform in m and n.

Note that a convergent single sequence is also almost convergent but for a double sequence this is not the case, that is, a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent and every almost convergent double sequence is bounded.

Using the idea of almost convergence, Lorentz [35] introduced and characterized strongly regular matrices.

We say that a four dimensional matrix A is strongly regular if every almost convergent double sequence x is A-summable to the same limit, and the A-means are also bounded.

If a double sequence x is almost convergent to L, then we write $f_2 - \lim x = L$ and f_2 for the space of almost convergent double sequences.

In [40], Moricz and Rhoades gave four dimensional analogue of strongly regular matrices as follows :

Theorem 6.2.1. Necessary and sufficient conditions for a matrix $A = (a_{jk}^{mn})$ to be strongly regular are that A is bounded-regular and satisfies the following two conditions :

$MR_1 :\ \lim_{m,n\to\infty} \sum_{j,k=0,0}^{\infty,\infty} \mid \Delta_{10}\ a_{jk}^{mn} \mid = 0;$

$MR_2 :\ \lim_{m,n\to\infty} \sum_{j,k=0,0}^{\infty,\infty} \mid \Delta_{01}\ a_{jk}^{mn} \mid = 0,$

where $\Delta_{10}\ a_{jk}^{mn} = a_{jk}^{mn} - a_{j+1,k}^{mn}$ and $\Delta_{01}\ a_{jk}^{mn} = a_{jk}^{mn} - a_{j,k+1}^{mn},\ (j,k = 0,1,\cdots).$

We quote here the following useful lemma :

Lemma 6.2.2 (Patterson [45]). If A is a real or complex-valued four dimensional matrix such that RH_3, RH_4, and

$$P-\limsup_{m,n}\sum_{j,k=0,0}^{\infty,\infty} \mid a_{jk}^{mn} \mid = M$$

hold, then for any bounded double sequence x we have

$$P-\limsup \mid Ax \mid \leq M(P-\limsup \mid x \mid) \ .$$

6.3. Main Result

We define the following :

Let us write

$$L^*(x) = \limsup_{p,q\to\infty}\sup_{m,n\geq 0}\frac{1}{pq}\sum_{j=m}^{m+p-1}\sum_{k=n}^{n+q-1} x_{jk} \ .$$

Then we define the MR-core of a real-valued bounded double sequence x to be the closed interval $[-L^*(-x), L^*(x)]$.

Since every bounded convergent double sequence is almost convergent, we have

$$L^*(x) \leq P-\limsup x = L(x), \ say,$$

and hence it follows that

$$MR\text{-core}\{x\} \subseteq P\text{-core}\{x\}$$

for a bounded double sequence $x = (x_{jk})_{j,k=0}^{\infty}$.

Here we prove a core theorem for double sequences making use of four dimensional strongly regular matrices due to Moricz and Rhoades [40].

Theorem 6.3.1. For every bounded double sequence x,

(6.3.1.1) $$L(Ax) \leq L^*(x)$$

(or P-core$\{Ax\} \subseteq MR$-core$\{x\}$) if and only if

(6.3.1.2) $A = (a_{jk}^{mn})$ is strongly regular;

(6.3.1.3) $P - \lim_{m,n\to\infty} \sum_{j,k=0,0}^{\infty,\infty} \mid a_{jk}^{mn} \mid = 1.$

Proof. Necessity. Let us consider a bounded double sequence x to be almost convergent to ℓ. Then we have $L^*(x) = -L^*(-x)$. By (6.3.1.1), we get

$$\ell = -L^*(-x) \leq -L(-Ax) \leq L(Ax) \leq L^*(x) = \ell \,.$$

Hence Ax is P-convergent and $P - \lim Ax = f_2 - \lim x = \ell$, and so A is strongly regular, i.e condition (6.3.1.2) holds.

Since every strongly regular matrix is also bounded-regular, by Lemma 6.2.2 there exists a bounded double sequence x such that $\limsup \mid x \mid = 1$ and $P - \limsup Ax = C$, where C is defined by RH_5. Therefore we have

$$1 \leq P - \liminf_{m,n} \sum_{j,k=0,0}^{\infty,\infty} \mid a_{jk}^{mn} \mid \leq P - \limsup_{m,n} \sum_{j,k=0,0}^{\infty,\infty} \mid a_{jk}^{mn} \mid \leq 1 \,,$$

i.e. condition (6.3.1.3) holds.

Sufficiency. Given $\varepsilon > 0$, we can find fixed integers $p, q \geq 2$ such that

(6.3.1.4) $$\frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} < L^*(x) + \varepsilon.$$

Now as in [40], we can write

(6.3.1.5) $$y_{MN} = \sum_{j,k=0,0}^{\infty,\infty} a_{jk}^{MN} \, x_{jk}$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 + \Sigma_7 + \Sigma_8 \,,$$

where

$$\sum_1 = \frac{1}{pq}\sum_{m,n=0,0}^{\infty,\infty} a_{mn}^{MN} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk}$$

$$\sum_2 = -\frac{1}{pq}\sum_{j=0}^{p-2} \sum_{k=0}^{q-2} x_{jk} \sum_{m=0}^{j} \sum_{n=0}^{k} a_{mn}^{MN}$$

$$\sum_3 = -\frac{1}{pq}\sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} x_{jk} \sum_{m=j-p+1}^{j} \sum_{n=0}^{k} a_{mn}^{MN}$$

$$\sum_4 = -\frac{1}{pq}\sum_{j=0}^{p-2} \sum_{k=q-1}^{\infty} x_{jk} \sum_{m=0}^{j} \sum_{n=k-q+1}^{k} a_{mn}^{MN}$$

$$\sum_5 = -\sum_{j=p-1}^{\infty} \sum_{k=q-1}^{\infty} x_{jk}\{\frac{1}{pq}\sum_{m=j-p+1}^{j} \sum_{n=k-q+1}^{k} a_{mn}^{MN} - a_{jk}^{MN}\}$$

$$\sum_6 = \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} a_{jk}^{MN} x_{jk}$$

$$\sum_7 = \sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} a_{jk}^{MN} x_{jk}$$

$$\sum_8 = -\sum_{j=0}^{p-2} \sum_{k=q-1}^{\infty} a_{jk}^{MN} x_{jk} .$$

Using the conditions of strong regularity of A, we observe that $(M, N \to \infty)$,

$$| \sum_2 | \leq \|x\|_{(\infty,2)} \sum_{m=0}^{p-2} \sum_{n=0}^{q-2} | a_{mn}^{MN} | \to 0 ,$$

and

$$| \sum_6 | \leq \|x\|_{(\infty,2)} \sum_{j=0}^{p-2}\sum_{k=0}^{q-2} | a_{jk}^{MN} | \to 0 , \text{ by } RH_1 ;$$

$$| \sum_3 | \leq \|x\|_{(\infty,2)} \sum_{m=0}^{\infty} \sum_{n=0}^{q-2} | a_{mn}^{MN} | \to 0 ,$$

and

$$| \sum_7 | \leq \|x\|_{(\infty,2)} \sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} | a_{jk}^{MN} | \to 0 , \text{ by } RH_3;$$

$$| \sum_4 | \to 0 \text{ and } | \sum_8 | \to 0 \text{ by } RH_4 .$$

Now

$$|\Sigma_5| \leq \frac{\|x\|_{(\infty,2)}}{pq} \sum_{r=0}^{p-1} \sum_{s=0}^{q-1}\{(p-r-1) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} | \Delta_{10}\, a_{jk}^{MN} |$$

$$+(q-s-1) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} | \Delta_{01}\, a_{jk}^{MN} |\} \to 0 \quad \text{by } MR_1 \text{ and } MR_2 .$$

Therefore we have by (6.3.1.5)

$$L(Ax) \leq \limsup_{M,N} \sum_{m,n=0,0}^{\infty,\infty} a_{mn}^{MN} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk}$$

$$\leq \limsup_{M,N} \Big| \sum_{m,n=0,0}^{\infty,\infty} \left(\frac{| a_{mn}^{MN} | + a_{mn}^{MN}}{2} + \frac{| a_{mn}^{MN} | - a_{mn}^{MN}}{2} \right) \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} \Big|$$

$$\leq \limsup_{M,N} \Big\{ \sum_{m,n=0,0}^{\infty,\infty} | a_{mn}^{MN} | \, \Big| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} \Big|$$

$$+\|x\|_{(\infty,2)} \sum_{m,n=0,0}^{\infty,\infty} \Big(| a_{mn}^{MN} | - a_{mn}^{MN} \Big) \Big\}.$$

Now conditions RH_1, RH_5 and (6.3.1.3) yield

$$L(Ax) \leq L^*(x) + \varepsilon.$$

Since ε is arbitrary we finally have

$$L(Ax) \leq L^*(x).$$

This completes the proof of the theorem.

6.4. Examples

6.4.1. Almost convergent sequences

(i) Define the double sequence $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} 1 & , \text{ if } j \text{ is odd, for all } k, \\ 0 & , \text{ otherwise.} \end{cases}$$

Then x is almost convergent to $\frac{1}{2}$.

(ii) Define $x = (x_{jk})$ by

$$x_{jk} = (-1)^j \text{ for all } k \ .$$

Then x is almost convergent to 0 .

6.4.2. Strongly regular matrix

Define $A = (a_{jk})$ by

$$a_{jk}^{mn} = \begin{cases} \frac{1}{m^2} & , \text{ if } m = n \text{ and } j, k \le m \text{ (even)}, \\ \\ \frac{1}{m^2 - m} & , \text{ if } m = n, \ j \ne k \text{ and } j, k \le m \text{ (odd)}, \\ \\ 0 & , \text{ otherwise.} \end{cases}$$

We can easily verify that A is strongly regular, that is, conditions $RH_1 - RH_5$, MR_1 and MR_2 hold. Moreover, for the sequence in 6.4.1(i), we have

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn} \ x_{jk} = a_{11}^{mm} \ x_{11} + a_{12}^{mm} \ x_{12} + \cdots + a_{1m}^{mm} \ x_{1m}$$

$$+ a_{21}^{mm} \ x_{21} + a_{22}^{mm} \ x_{22} + \cdots + a_{2m}^{mm} \ x_{2m}$$

$$+ a_{31}^{mm}\, x_{31} + a_{32}^{mm}\, x_{32} + a_{33}^{mm}\, x_{33} + \cdots + a_{3m}^{mm}\, x_{3m}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ a_{m-1,1}^{mm}\, x_{m-1,1} + \cdots\cdots + a_{m-1,m}^{mm}\, x_{m-1,m}$$

$$+ a_{m1}^{mm}\, x_{m1} + \cdots\cdots + a_{mm}^{mm}\, x_{mm}$$

$$= \frac{m}{m^2} \cdot \frac{m}{2}, \quad \text{if } m \text{ is even},$$

$$\to \frac{1}{2} \quad \text{as} \quad m, n \to \infty\ .$$

Similarly

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn}\, x_{jk} = \frac{m-1}{m^2 - m} \cdot \frac{m+1}{2} \quad \text{if } m \text{ is odd},$$

$$\to \frac{1}{2} \quad \text{as} \quad m, n \to \infty\ .$$

That is

$$P - \lim Ax = \frac{1}{2} = f_2 - \lim x\ ,$$

and so A transforms almost convergent sequence into convergent (P-convergent) to the same limit.

6.4.3. Bounded-regular matrix which is not strongly regular

In 6.4.2, A is strongly regular and so bounded regular. Let us define $A = (a_{jk}^{mn})$ as

$$a_{jk}^{mn} = \begin{cases} \frac{2}{m^2} & , \text{ if } m = n, \ j + k = \text{even}, \quad \text{and } \ j, k \leq m \text{ (even)}, \\ \\ \frac{1}{m^2 - m} & , \text{ if } m = n, \ j \neq k \ \text{ and } \ j, k \leq m \text{ (odd)}, \\ \\ 0 & , \text{ otherwise.} \end{cases}$$

Then A is bounded-regular but not strongly regular. Conditions $RH_1 - RH_5$ can easily be verified. But

$$\lim_{m,n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mid a_{jk}^{mn} - a_{j+1,k}^{mn} \mid = \begin{cases} 2 & , \ \text{if } m \text{ is even,} \\ 0 & , \ \text{if } m \text{ is odd,} \end{cases}$$

and also

$$\lim_{m,n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mid a_{jk}^{mn} - a_{j,k+1}^{mn} \mid = \begin{cases} 2 & , \ \text{if } m \text{ is even,} \\ 0 & , \ \text{if } m \text{ is odd.} \end{cases}$$

Therefore conditions MR_1 and MR_2 do not hold and so A is not strongly regular.

6.4.4. In Theorem 6.3.1, strong regularity of A can not be replaced by bounded-regularity.

Consider the matrix $A = (a_{jk}^{mn})$ as defined in 6.4.3. This is bounded-regular but not strongly regular, and also

$$P - \lim_{m,n} \sum_{j,k=0,0}^{\infty,\infty} \mid a_{jk}^{mn} \mid = 1 \ ,$$

i.e condition (6.3.1.3) of Theorem 6.3.1 holds. Take the bounded double sequence $x = (x_{jk})$ defined by $x_{jk} = (-1)^{j+k}$, which is almost convergent to zero, that is, $L^*(x) = 0$.

Now

$$\sum_{j,k} a_{jk}^{mn}\, x_{jk} = \begin{cases} \frac{2}{m^2} \cdot \frac{m}{2} \cdot m & , \text{ if } m \text{ is even,} \\ \\ \frac{-1}{m^2-m} \cdot m & , \text{ if } m \text{ is odd .} \end{cases}$$

Therefore

$$\limsup_{m,n} \sum_{j,k} a_{jk}^{mn}\, x_{jk} = 1$$

and

$$\liminf_{m,n} \sum_{j,k} a_{jk}^{mn}\, x_{jk} = 0$$

i.e. $L(Ax) = 1$. Hence $L(Ax) > L^*(x)$, that is (6.3.1.1) does not hold.

CHAPTER VII

STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

7.1. Introduction

Two dimensional analogue of natural density have been introduced by Christopher [5] which we will use to define statistical convergence of double sequences.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K(n,m)$ be the numbers (i,j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of K is defined as

$$\liminf_{n,m} \frac{K(n,m)}{nm} = \underline{\delta_2}(K).$$

In case the sequence $\left(\frac{K(n,m)}{nm}\right)$ has a limit then we say that K has a double natural density and is defined as

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\,\sqrt{m}}{nm} = 0,$$

i.e. the set K has double natural density zero.

In this chapter we define and study statistical analogue of convergence and Cauchy for double sequences using the idea of double natural density due to Christopher [5]. We also establish the relation between statistical convergence and strongly Cecàro summable sequences.

7.2. Statistical Convergence

We define the statistical analogue for double sequences $x = (x_{jk})$ as follows:

Definition 7.2.1. A real double sequence $x = (x_{jk})$ is said to be statistically convergent to the number ℓ if for each $\varepsilon > 0$, the set

$$\{(j,k), j \leq n \text{ and } k \leq m : |\, x_{jk} - \ell \,| \geq \varepsilon\}$$

has double natural density zero. In this case we write $st_2 - \lim_{n,m} x_{nm} = \ell$ and we denote the set of all statistically convergent double sequences by st_2.

Remark 7.2.2. (a) If x is a convergent double sequence then it is also statistically convergent to the same number. As if x is bounded then the set

$$\{(j,k), j \leq n,\ k \leq m : |\, x_{jk} - \ell \,| \geq \varepsilon\}$$

is finite for each $\varepsilon > 0$ and hence of natural density zero.

In case x is convergent but not bounded, then there are only a finite number of unbounded rows and (or) columns and hence

$$K(n,m) \leq s_1 m + s_2 n$$

where s_1 and s_2 are finite numbers, which we can conclude that x is statistically convergent.

(b) If x is statistically convergent to the number ℓ, then ℓ is determined uniquely.

(c) If x is statistically convergent, then x need not be convergent. Also it is not necessary bounded. For example, let $x = (x_{jk})$ be defined as

$$x_{jk} = \begin{cases} jk & , \text{ if } j \text{ and } k \text{ are squares,} \\ 1 & , \text{ otherwise.} \end{cases}$$

It is easy to see that $st_2 - \lim x_{jk} = 1$, since

$$\delta_2\{(j,k) : x_{jk} \neq 1\} \leq \lim_{j,k} \frac{\sqrt{j}\,\sqrt{k}}{jk} = 0.$$

But x is neither convergent nor bounded.

We prove some analogues for double sequences. For single sequences such results have been proved by Šalát [49].

Theorem 7.2.3. A real double sequence $x = (x_{jk})$ is statistically convergent to a number ℓ if and only if there exists a subset $K = \{(i,s)\} \subseteq I\!N \times I\!N$, $i, s = 1, 2, \cdots$ such that $\delta(K) = 1$ and

$$\lim_{i,s} x_{j_i,k_s} = \ell.$$

Proof. Let x be statistically convergent to ℓ. Then for every $\varepsilon > 0$ the set

$$K = \{(j,k) : | x_{jk} - \ell | \geq \varepsilon\}$$

has natural density zero and hence its complement

$$M = \{(i,s) : | x_{is} - \ell | < \varepsilon\}$$

has natural density 1, i.e.

$$\delta_2(M) = \delta_2(I\!N \times I\!N) - \delta_2(K) = 1 - 0 = 1,$$

and

$$K \cap M = \emptyset.$$

Now we have to show that (x_{is}), $i, s \in M$ is convergent to ℓ. Suppose that (x_{is}) is not convergent to ℓ. Then there exists $\varepsilon_o > 0$ such that

$$| x_{is} - \ell | \geq \varepsilon_o \text{ for infinitely many terms.}$$

Let

$$M' = \{(i', s'), i' \leq i, s' \leq s : | x_{i's'} - \ell | \geq \varepsilon_o\}.$$

Clearly $\emptyset \neq M' \subseteq M$. Then

$$K = \{(j,k) : | x_{jk} - \ell | \geq \varepsilon_o\} \supseteq \{(i', s') : | x_{i's'} - \ell | \geq \varepsilon_o\}.$$

Hence $\delta_2(M') = 0$, i.e. $M' \subseteq K$ which contradicts the fact that $K \cap M = \emptyset$. Hence (x_{is}) is convergent to ℓ.

Conversely, Suppose that there exists a subset $K = \{(j_i, k_s)\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $\lim_{i,s} x_{j_i,k_s} = \ell$, i.e. there exists $N \in \mathbb{N}$ such that

$$| x_{j_i,k_s} - \ell | < \varepsilon, \ \ \forall \ j_i, k_s > N.$$

Now

$$\{(j,k) : | x_{jk} - \ell | \geq \varepsilon\} \subseteq \mathbb{N} \times \mathbb{N} - \{(j_{N+1}, k_{N+1}), (j_{N+2}, k_{N+2}), \cdots\}.$$

Therefore

$$\delta_2\{(j,k) : | x_{jk} - \ell | \geq \varepsilon\} \leq 1 - 1 = 0.$$

Hence x is statistically convergent to ℓ.

Theorem 7.2.4. The set $st_2 \cap \ell^2_\infty$ is a closed linear subspace of the normed linear space ℓ^2_∞.

Proof. Let $x^{(nm)} = (x^{(nm)}_{jk}) \in st_2 \cap \ell^2_\infty$ and $x^{(nm)} \to x \in \ell^2_\infty$. Since $x^{(nm)} \in st_2 \cap \ell^2_\infty$, there exists a real number a_{nm} such that

$$st_2 - \lim_{j,k} x^{(nm)}_{jk} = a_{nm} \ \ (n, m = 1, 2, \cdots).$$

Also as $x^{(nm)} \to x$. Therefore for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$| x^{(pq)} - x^{(nm)} | < \varepsilon/3 \tag{7.2.4.1}$$

for every $p \geq n \geq N, \ q \geq m \geq N$.

From Theorem 7.2.3, there exist subsets K_{pq} and K_{nm} of $\mathbb{N} \times \mathbb{N}$ with $\delta_2(K_{pq}) = \delta_2(K_{nm}) = 1$ and

(1)$\cdots\cdots$
$$\lim_{j,k} {}_{j,k \in K_{nm}} x^{(nm)}_{jk} = a_{nm},$$

(2)$\cdots\cdots$
$$\lim_{j,k} {}_{j,k \in K_{pq}} x^{(pq)}_{jk} = a_{pq}.$$

Now the set $K_{pq} \cap K_{nm}$ is infinite since $\delta_2(K_{pq} \cap K_{nm}) = 1$.

Choose $(k_1, k_2) \in K_{pq} \cap K_{nm}$. We have from (1) and (2) that,

$$(7.2.4.2) \qquad \mid x^{(pq)}_{k_1,k_2} - a_{pq} \mid < \varepsilon/3,$$

and

$$(7.2.4.3) \qquad \mid x^{(nm)}_{k_1,k_2} - a_{nm} \mid < \varepsilon/3.$$

Therefore for each $p \geq n \geq N$ and $q \geq m \geq N$ we have from (7.2.4.1), (7.2.4.2) and (7.2.4.3)

$$\begin{aligned} \mid a_{pq} - a_{nm} \mid &\leq \mid a_{pq} - x^{pq}_{k_1,k_2} \mid + \mid x^{pq}_{k_1,k_2} - x^{nm}_{k_1,k_2} \mid + \mid x^{nm}_{k_1,k_2} - a_{nm} \mid \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

That is the sequence (a_{nm}) is a Cauchy sequence and hence convergent. Let

$$(3) \cdots\cdots \qquad \lim_{n,m} a_{nm} = a.$$

We need to show that x is statistically convergent to a. Since $x^{(nm)}$ is convergent to x, there exists $N_o \in \mathbb{N}$ such that for every $\varepsilon > 0$ and $n, m \geq N_o$

$$\mid x^{(nm)} - x \mid < \varepsilon/3.$$

Also from (3) we have for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$

$$\mid a_{nm} - a \mid < \varepsilon/3.$$

Again, since $x^{(nm)}$ is statistically convergent to a_{nm}, there exists a set $K_{jk} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K_{jk}) = 1$ and for every $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$

$$\mid x^{(nm)}_{jk} - a_{nm} \mid < \varepsilon/3, \quad \forall\ j, k \in K_{jk}.$$

Let $max\{N_o, N_1, N_2\} = N_3$. Then for a given $\varepsilon > 0$, for every $j, k \in K_{jk}$ and for all $n, m \geq N_3$

$$\begin{aligned} | x_{jk} - a | &\leq | x_{jk} - x_{jk}^{(nm)} | + | x_{jk}^{(nm)} - a_{nm} | + | a_{nm} - a | \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Therefore x is statistically convergent to a, i.e. $x \in st_2 \cap \ell^2_\infty$.

Hence $st_2 \cap \ell^2_\infty$ is a closed linear subspace of ℓ^2_∞.

Theorem 7.2.5. The set $st_2 \cap \ell^2_\infty$ is nowhere dense in ℓ^2_∞.

Proof. Since every closed linear subspace of an arbitrary linear normed space S different from S is a nowhere dense set in S (cf. Neubrum, Smital and Šalát [43]). From Theorem 7.2.4 we need only to show that $st_2 \cap \ell^2_\infty \neq \ell^2_\infty$.

Let the sequence $x = (x_{jk})$ be defined by

$$x_{jk} = \begin{cases} 1 & , \text{ if } j \text{ and } k \text{ are even,} \\ 0 & , \text{ otherwise.} \end{cases}$$

It is clear that x is not statistically convergent but x is bounded. Hence $st_2 \cap \ell^2_\infty \neq \ell^2_\infty$.

7.3. Statistically Cauchy Sequences

In [18], Fridy has defined the concept of statistically Cauchy single sequences. In this section we define statistically Cauchy double sequences and prove some analogues.

Definition 7.3.1. A real double sequence $x = (x_{jk})$ is said to be statistically Cauchy if for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that the set

$$\{(j,k), j \leq n, k \leq m : | x_{jk} - x_{NM} | \geq \varepsilon\}$$

has double natural density zero.

Theorem 7.3.2. A real double sequence $x = (x_{jk})$ is statistically convergent if and only if x is statistically Cauchy.

Proof. Let x be statistically convergent to a number ℓ. Then for every $\varepsilon > 0$, the set

$$\{(j,k), j \leq n, k \leq m : |\, x_{jk} - \ell \,| \geq \varepsilon\}$$

has natural density zero. We can choose two numbers N and M such that $|\, x_{NM} - \ell \,| \geq \varepsilon$.

Now

$$A = \{(j,k), j \leq n, k \leq m : |\, x_{jk} - x_{NM} \,| \geq \varepsilon\} \subseteq B \cup C,$$

where

$$B = \{(j,k), j \leq n, k \leq m : |\, x_{jk} - \ell \,| \geq \varepsilon\}$$

and

$$C = \{(j,k), j \leq n, k \leq m : |\, x_{NM} - \ell \,| \geq \varepsilon\}.$$

Therefore $\delta_2(A) \leq \delta_2(B) + \delta_2(C) = 0$. Hence x is statistically Cauchy.

Conversely, let x be statistically Cauchy but not statistically convergent. Then there exist N and M such that the set

$$K_{NM} = \{(j,k), j \leq n, k \leq m : |\, x_{jk} - x_{NM} \,| \geq \varepsilon\}$$

has natural density zero. Hence the set

$$E_{NM} = \{(i,s), i \leq n, s \leq m : |\, x_{is} - x_{NM} \,| < \varepsilon\}$$

has natural density 1. Also

$$E_{NM} \cap K_{NM} = \emptyset.$$

Now, since x is not statistically convergent and $\delta(E_{NM}) = 1$ so that from Theorem 7.2.3, the subsequence (x_{is}); $i, s \in E_{NM}$ does not converge to any number. Therefore there exists $\varepsilon_o > 0$ such that for all $N_o \in \mathbb{N}$

$$| x_{pq} - x_{is} | \geq \varepsilon_o, \ \forall\, p \geq i \geq N_o, \ q \geq s \geq N_o.$$

Now, consider the set

$$E'_{NM} = \{(p, q) : | x_{pq} - x_{NM} | \geq \varepsilon_o\} \neq \emptyset.$$

Then

$$\{(j, k) : | x_{jk} - x_{NM} | \geq \varepsilon_o\} \supseteq \{(p, q) : | x_{pq} - x_{NM} | \geq \varepsilon_o\}.$$

This implies that $\delta_2(E'_{NM}) = 0$, i.e. $E'_{NM} \subseteq K_{NM}$ which is a contradiction. Hence x is statistically convergent.

This completes the proof of the theorem.

From Theorems 7.2.3 and 7.3.2 we can state the following for double sequences analogous to the result of Fridy [18].

Theorem 7.3.3. The following statements are equivalent:

(7.3.3.1) x is statistically convergent

(7.3.3.2) x is statistically Cauchy

(7.3.3.3) there exists a set $K_{nm} = \{(j_1, k_1), \cdots, (j_n, k_m)\}$ such that $\delta_2(K_{nm}) = 1$ and

$\lim_{n,m} x_{j_n, k_m} = 1.$

Corollary 7.3.4. If x is statistically convergent to ℓ then there exists a subsequence y of x such that

$$\lim y = \ell \ \text{ and } \ \delta_2\{(j, k) : x_{jk} = y_{jk}\} = 1.$$

7.4. Relation Between Statistical Convergence and Strongly Cesàro Summable Sequences

In [39], Moricz defined the means C_{11}, C_{10} and C_{01} of $x = (x_{jk})$ respectively by

$$\sigma_{mn}^{11} = \frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} x_{jk},$$

$$\sigma_{mn}^{10} = \frac{1}{m}\sum_{j=1}^{m} x_{jn},$$

and

$$\sigma_{mn}^{01} = \frac{1}{n}\sum_{k=1}^{n} x_{mk}.$$

We say that a double sequence $x = (x_{jk})$ is C_{11}-summable or Cesàro summable to a finite limit ℓ if the sequence (σ_{mn}^{11}) is convergent to ℓ in Pringsheim's sense, i.e.

$$\lim_{n,m}\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} x_{jk} = \ell.$$

Similarly C_{10} and C_{01} summable sequences are defined.

We can define the following as in case of single sequences.

Definition 7.4.1. Let $x = (x_{jk})$ be a double sequence and p be a positive real number. Then the double sequence x is said to be strongly p-Cesàro summable to ℓ if

$$\lim_{n,m}\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} | x_{jk} - \ell |^{p} = 0.$$

We denote the space of all strongly p-Cesàro summable double sequences by w_p^2.

Remark 7.4.2. (i) If $0 \leq p \leq q < \infty$, then $w_q^2 \subseteq w_p^2$ (by Hölder's inequality) and

$$w_p^2 \cap \ell_\infty^2 = w_1^2 \cap \ell_\infty^2 \subseteq C_{11} \cap \ell_\infty^2.$$

(ii) If x is convergent but unbounded then x is statistically convergent but x need not be Cesàro nor strongly Cesàro.

Example 1. Let $x = (x_{jk})$ be defined as

$$x_{jk} = \begin{cases} k & , \; j = 1, \text{ for all } k, \\ j & , \; k = 1, \text{ for all } j, \\ 1 & , \text{ otherwise.} \end{cases}$$

Then $\lim_{j,k} x_{jk} = 1$ but

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = \lim_{n,m} \frac{1}{nm} \Big[(\frac{1}{2}m + \frac{1}{2}m^2) + (m+1) + \cdots + (m + (n-1)) \Big]$$

$$= \lim_{n,m} \frac{1}{nm} \Big[\frac{1}{2}m + \frac{1}{2}m^2 + (n-1)m + \frac{1}{2}n^2 - \frac{1}{2}n \Big]$$

which does not tend to a finite limit. Hence x is not Cesàro. Also x is not strongly Cesàro but

$$\lim_{n,m} \frac{1}{nm} \mid \{(j,k) : \mid x_{jk} - 1 \mid \geq \varepsilon\} \mid = \lim_{n,m} \frac{m+n-1}{nm} = 0,$$

i.e. x is statistically convergent to 1.

(iii) If x is a bounded convergent double sequence then it is also C_{11}, w_p^2 and st_2.

The following result is analogue of Theorem 2.1 [6].

Theorem 7.4.3. Let $x = (x_{jk})$ be a double sequence and p be a positive real number. Then

(a) if x is strongly p-Cesàro summable to ℓ, then it is also statistically convergent to ℓ,

(b) $w_p^2 \cap \ell_\infty^2 = st_2 \cap \ell_\infty^2$.

Proof. (a) Let

$$K_{nm} = \{(j,k), j \le n, k \le m : | x_{jk} - \ell |^p \ge \varepsilon\}.$$

Now since x is strongly p-Cesàro summable to ℓ then

$$0 \leftarrow \frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} | x_{jk} - \ell |^p = \frac{1}{nm}\Big\{ \textstyle\sum_{j\in K_{nm}} \sum_{k\in K_{nm}} | x_{jk} - \ell |^p$$

$$+ \textstyle\sum_{j\notin K_{nm}} \sum_{k\notin K_{nm}} | x_{jk} - \ell |^p + \sum_{j\in K_{nm}} \sum_{k\notin K_{nm}} | x_{jk} - \ell |^p$$

$$+ \textstyle\sum_{j\notin K_{nm}} \sum_{k\in K_{nm}} | x_{jk} - \ell |^p \Big\}$$

$$\ge \frac{1}{nm}\Big\{ \textstyle\sum_{j\in K_{nm}} \sum_{k\in K_{nm}} | x_{jk} - \ell |^p + \sum_{j\notin K_{nm}} \sum_{k\notin K_{nm}} | x_{jk} - \ell |^p \Big\}$$

$$\ge \frac{1}{nm} | \{(j,k), j \le n, k \le m : | x_{jk} - \ell |^p \ge \varepsilon\} | \varepsilon^{1/p}.$$

Hence x is statistically convergent to ℓ.

(b) Let

$$I_{nm} = \Big\{ (j,k), j \le n, k \le m : | x_{jk} - \ell | \ge (\tfrac{\varepsilon}{4})^{1/p} \Big\}$$

and $M = \|x\|_{(\infty,2)} + | \ell |$, where $\|x\|_{(\infty,2)}$ is the sup-norm for bounded double sequences $x = (x_{jk})$.

Since x is a bounded statistically convergent, we can choose $N = N(\varepsilon)$ such that for all $n, m > N$

$$\frac{1}{nm} | \{(j,k), j \le n, k \le m : | x_{jk} - \ell | \ge (\frac{\varepsilon}{4})^{1/p}\} | < \frac{\varepsilon}{4M^p} .$$

Now for all $n, m > N$ we have

$$\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} | x_{jk} - \ell |^p = \frac{1}{nm}\Big\{ \textstyle\sum_{j\in I_{nm}} \sum_{k\in I_{nm}} | x_{jk} - \ell |^p$$

$$+ \sum_{j\notin I_{nm}} \sum_{k\notin I_{nm}} | x_{jk} - \ell |^p + \sum_{j\in I_{nm}} \sum_{k\notin I_{nm}} | x_{jk} - \ell |^p$$

$$+ \textstyle\sum_{j\notin I_{nm}} \sum_{k\in I_{nm}} | x_{jk} - \ell |^p \Big\}$$

$$< \frac{1}{nm}\, nm\, \frac{\varepsilon}{4M^p} K^p + \frac{1}{nm}\, nm\, \frac{\varepsilon}{4} + \frac{1}{nm}\, nm\, \frac{\varepsilon}{4} + \frac{1}{nm}\, nm\, \frac{\varepsilon}{4}$$

$$= \varepsilon.$$

Hence x is strongly p-Cesàro summable to ℓ.

Remark 7.4.4. Note that if a bounded sequence x is statistically convergent then it is also C_{11} summable but not conversely.

Example 2. Let $x = (x_{jk})$ be defined by

$$x_{jk} = (-1)^j, \;\; \forall\, k$$

then

$$\lim_{n,m}\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} x_{jk} = 0,$$

but obviously x is not statistically convergent.

CHAPTER VIII

TAUBERIAN THEOREMS FOR STATISTICALLY CONVERGENT DOUBLE SEQUENCES

8.1. Introduction

In [39], Moricz has proved some Tauberian theorems for Cesàro summable double sequences and deduced the Tauberian theorems of Landau [33] and Hardy [26] type. Friday and Khan [20] have proved statistical extensions of such classical Tauberian theorems.

Let (x_k) be a sequence of real numbers. Landau [33] gave a classical one-side Tauberian theorem as follows:

Lemma 8.1.1. If (x_k) is summable C_1 to a finite number ℓ and there exists a constant M such that

$$(*) \qquad k(x_k - x_{k-1}) \geq -M \quad (k = 1, 2, \cdots)$$

then (x_k) converges to ℓ.

We say that (x_k) is slowly decreasing (cf. Schmidt [50]) if for each $\varepsilon > 0$ there exist $n_1 > 0$ and $\lambda > 1$ such that

$$(**) \qquad x_k - x_n \geq -\varepsilon \quad \text{whenever} \quad n_1 < n < k \leq \lambda n.$$

Clearly $(*)$ is a particular case of $(**)$.

Lemma 8.1.2 (Hardy [26]). If a sequence (x_k) is summable C_1 to a finite number ℓ and (x_k) is slowly decreasing, then (x_k) converges to ℓ.

Landau's theorem remains valid if condition $(*)$ is replaced by (cf. Zygmund [55])

$$k \mid \nabla x_k \mid \leq H \quad (k = 1, 2, \cdots)$$

where the "backward difference" $\nabla x_k = x_k - x_{k-1}$ with $x_{-1} = 0$.

Moricz [39] have proved the following results for double sequences.

Theorem 8.1.3. If (x_{jk}) is summable C_{11} to a finite limit and there exist constants n_1 and M such that the following conditions hold:

(8.1.3.1) $jk(x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}) \geq -M$ whenever $j, k > n_1$;

(8.1.3.2) $j(x_{jk} - x_{j-1,k}) \geq -M$ whenever $j, k > n_1$;

(8.1.3.3) $k(x_{jk} - x_{j,k-1}) \geq -M$ whenever $j, k > n_1$.

Then (x_{jk}) converges.

Theorem 8.1.4. If (x_{jk}) is summable C_{11} to a finite limit and there exist constants n_1 and M such that the following conditions hold:

(8.1.4.1) $jk \mid x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1} \mid \leq M$ whenever $j, k > n_1$;

(8.1.4.2) $j \mid x_{jk} - x_{j-1,k} \mid \leq M$ whenever $j, k > n_1$,

(8.1.4.3) $k \mid x_{jk} - x_{j,k-1} \mid \leq M$ whenever $j, k > n_1$.

Then (x_{jk}) converges.

Fridy [18] established a Tauberian theorem for statistical convergence of single sequences.

Theorem 8.1.5. If $st - \lim x = \ell$ and $\nabla x_k = O(\frac{1}{k})$, then $\lim x = \ell$.

In [20], Fridy and Khan have extended this idea as follows:

Theorem 8.1.6. If $st-\lim C_1\ x=\ell$ and $\nabla x_k = O(\frac{1}{k})$, then $\lim x = \ell$.

Lemma 8.1.7. If $\nabla x_k = O(\frac{1}{k})$, then $(\nabla C_1\ x)_n = O(\frac{1}{n})$.

Theorem 8.1.8. If $st-\lim C_1\ x=\ell$ and $\nabla x_k = O(\frac{1}{k})$, then $\lim x = \ell$.

Theorem 8.1.9. If $st-\lim x=\ell$ and $k\nabla x_{k+1} \geq -c$, for some $c>0$ and for every k, then $\lim x = \ell$.

Theorem 8.1.10. If $st-\lim C_1\ x=\ell$ and $k\nabla x_{k+1} \geq -c$, for some $c>0$ and for every k, then $\lim x = \ell$.

In this chapter we obtain Tauberian results for statistically convergent double sequences.

8.2. Main Results

Here we denote the backward differences of x_{jk} as follows:

$$\nabla_{11}\ x_{jk} = x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}$$

$$\nabla_{10}\ x_{jk} = x_{jk} - x_{j-1,k}$$

$$\nabla_{01}\ x_{jk} = x_{jk} - x_{j,k-1}.$$

We can also easily compute the following differences:

D_1. $$x_{jk} - x_{j+p,k+q} = \sum_{i=j+1}^{j+p} \sum_{s=k+1}^{k+q} \nabla_{11}\ x_{is}$$

$$- \sum_{i=j+1}^{j+p} \nabla_{10}\ x_{i,k+q} - \sum_{s=k+1}^{k+q} \nabla_{01}\ x_{j+p,s}$$

$$= - \sum_{i=j+1}^{j+p} \nabla_{10}\ x_{ik} - \sum_{s=k+1}^{k+q} \nabla_{01}\ x_{j+p,s}$$

D_2. $\quad x_{nj} - x_{nk} = \sum_{s=k+1}^{j} \nabla_{01}\, x_{ns}$

D_3. $\quad x_{nm} - x_{km} = \sum_{i=k+1}^{n} \nabla_{10}\, x_{im}$

D_4. $\quad x_{jk} - x_{nm} = \sum_{i=j+1}^{n} \sum_{s=k+1}^{m} \nabla_{11}\, x_{is}$

$$- \sum_{i=j+1}^{n} \nabla_{10}\, x_{im} - \sum_{s=k+1}^{m} \nabla_{01}\, x_{ns}$$

D_5. $\quad x_{jk} - x_{nm} - x_{nk} + x_{nm} = \sum_{i=j+1}^{n} \sum_{s=k+1}^{m} \nabla_{11}\, x_{is}$.

Our first Tauberian theorem is as follows:

Theorem 8.2.1. Let $x = (x_{jk})$ be a double sequence with $st_2 - \lim x = \ell$ and there exist a constant n_1 such that

(8.2.1.1) $\quad \nabla_{11}\, x_{jk} = O(\frac{1}{jk})$;

(8.2.1.2) $\quad \nabla_{10}\, x_{jk} = O(\frac{1}{j})$;

(8.2.1.3) $\quad \nabla_{01}\, x_{jk} = O(\frac{1}{k})$,

whenever $j, k > n_1$. Then $\lim_{j,k} x_{jk} = \ell$.

Proof. Since $st_2 - \lim x = \ell$, by Corollary 7.3.4, there exists a sequence $y = (y_{jk})$ such that $\lim_{j,k} y_{jk} = \ell$ and

(8.2.1.4) $$\delta_2\{(j,k) : j \le n, k \le m : y_{jk} = x_{jk}\} = 1.$$

For each j and k we write

$$j = \lambda(j) + \mu(j) \quad \text{and} \quad k = \lambda(k) + \mu(k),$$

where $\lambda(j) = max\{i \le j : y_{ik} = x_{ik}\}$ and $\lambda(k) = max\{s \le k : y_{js} = x_{js}\}$. In case the sets $\{i \le j : y_{ik} = x_{ik}\}$ and / or $\{s \le k : y_{js} = x_{js}\}$ are empty, we take $\lambda(j) = -1$, $\lambda(k) = -1$. Note that this can happen only for a finite number of j

and k.

Now we will show that

$$\lim_j \frac{\mu(j)}{\lambda(j)} = 0 = \lim_k \frac{\mu(k)}{\lambda(k)} \; . \tag{8.2.1.5}$$

Suppose that

$$\frac{\mu(j)}{\lambda(j)} > \varepsilon > 0 \quad \text{and} \quad \frac{\mu(k)}{\lambda(k)} > \varepsilon > 0. \tag{8.2.1.6}$$

Then

$$\frac{1}{jk} \mid \{(i,s), i \leq j, s \leq k : x_{is} \neq y_{is}\} \mid$$

$$\leq \frac{\lambda(j)\mu(k) + \lambda(k)\mu(j) + \mu(j)\mu(k)}{(\lambda(j)+\mu(j))(\lambda(k)+\mu(k))}$$

$$= \frac{\lambda(j)\mu(k) + \lambda(k)\mu(j) + \mu(j)\mu(k)}{\lambda(j)\lambda(k) + \lambda(j)\mu(k) + \mu(j)\lambda(k) + \mu(j)\mu(k)}$$

$$\leq \frac{\varepsilon + \varepsilon + \varepsilon^2}{\varepsilon^2 + \varepsilon + \varepsilon + 1} = \frac{\varepsilon^2 + 2\varepsilon}{\varepsilon^2 + 2\varepsilon + 1} \text{ by } (8.2.1.6),$$

i.e. if (8.2.1.6) holds for infinitely many terms j and k we get a contradiction to (8.2.1.4).

Hence (8.2.1.5) holds.

Now by the conditions (8.2.1.1) $-$ (8.2.1.3) and D_1, we have

$$\mid y_{\lambda(j),\lambda(k)} - x_{jk} \mid = \mid x_{\lambda(j),\lambda(k)} - x_{\lambda(j)+\mu(j),\lambda(k)+\mu(k)} \mid$$

$$\leq \sum_{i=\lambda(j)+1}^{\lambda(j)+\mu(j)} \sum_{s=\mu(k)+1}^{\lambda(k)+\mu(k)} \mid \nabla_{11}\, x_{is} \mid + \sum_{i=\lambda(j)+1}^{\lambda(j)+\mu(j)} \mid \nabla_{10}\, x_{i,\lambda(k)+\mu(k)} \mid$$

$$+ \sum_{s=\mu(k)+1}^{\lambda(k)+\mu(k)} \mid \nabla_{01} x_{\lambda(j)+\mu(j),s} \mid$$

$$\leq \frac{A\lambda(j)\mu(k)}{(\lambda(j)+1)(\lambda(k)+1)} + \frac{B\mu(j)}{\lambda(j)+1} + \frac{C\mu(k)}{\mu(k)+1}$$

$$\to 0 \text{ as } j,k \to \infty \text{ by } (8.2.1.5) ,$$

where A, B and C are constants.

Hence $\lim_{j,k} x_{jk} = \ell$, since $\lim_{j,k} y_{jk} = \ell$.

Lemma 8.2.2. Let $x = (x_{jk})$ be a double sequence. If there exists a constant n_1 such that the following conditions hold

(8.2.2.1) $\nabla_{11}\ x_{jk} = O(\frac{1}{jk})$;

(8.2.2.2) $\nabla_{10}\ x_{jk} = O(\frac{1}{j})$;

(8.2.2.3) $\nabla_{01}\ x_{jk} = O(\frac{1}{k})$,

whenever $j, k > n_1$. Then $(\nabla_{11}\ C_{11}\ x)_{nm} = O(\frac{1}{nm})$.

Proof. For all $n, m > 1$

$$nm(\nabla_{11}\ C_{11}\ x)_{nm} = nm\Bigg[\frac{1}{nm}\sum_{j=1}^{n}\sum_{k=1}^{m} x_{jk} - \frac{1}{(n-1)m}\sum_{j=1}^{n-1}\sum_{k=1}^{m} x_{jk}$$

$$- \frac{1}{n(m-1)}\sum_{j=1}^{n}\sum_{k=1}^{m-1} x_{jk} + \frac{1}{(n-1)(m-1)}\sum_{j=1}^{n-1}\sum_{k=1}^{m-1} x_{jk}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\Bigg[(n-1)(m-1)\sum_{j=1}^{n}\sum_{k=1}^{m} x_{jk}$$

$$-n(m-1)\sum_{j=1}^{n-1}\sum_{k=1}^{m} x_{jk} - m(n-1)\sum_{j=1}^{n}\sum_{k=1}^{m-1} x_{jk} + nm\sum_{j=1}^{n-1}\sum_{k=1}^{m-1} x_{jk}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\Bigg[nm\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk} - n\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk} - m\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk}$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk} - nm\sum_{j=1}^{n-1}\sum_{k=1}^{m}x_{jk} + n\sum_{j=1}^{n-1}\sum_{k=1}^{m}x_{jk}$$

$$-nm\sum_{j=1}^{n}\sum_{k=1}^{m-1}x_{jk} + m\sum_{j=1}^{n}\sum_{k=1}^{m-1}x_{jk} + nm\sum_{j=1}^{n-1}\sum_{k=1}^{m-1}x_{jk}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\Bigg[nm\Big(\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk} - \sum_{j=1}^{n-1}\sum_{k=1}^{m}x_{jk} - \sum_{j=1}^{n}\sum_{k=1}^{m-1}x_{jk} + \sum_{j=1}^{n-1}\sum_{k=1}^{m-1}x_{jk}\Big)$$

$$-n\Big(\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk} - \sum_{j=1}^{n-1}\sum_{k=1}^{m}x_{jk}\Big)$$

$$-m\Big(\sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk} - \sum_{j=1}^{n}\sum_{k=1}^{m-1}x_{jk}\Big) + \sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\Bigg[nm\ x_{nm} - n\sum_{k=1}^{m}x_{nk} - m\sum_{j=1}^{n}x_{jm} + \sum_{j=1}^{n}\sum_{k=1}^{m}x_{jk}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\Bigg[nm\ x_{nm} - n\sum_{k=1}^{m-1}x_{nk} - n\ x_{nm} - m\sum_{j=1}^{n-1}x_{jm} - m\ x_{nm}$$

$$+\sum_{j=1}^{n-1}\sum_{k=1}^{m-1}x_{jk} + \sum_{j=1}^{n-1}x_{jm} + \sum_{k=1}^{m-1}x_{nk} + x_{nm}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\Bigg[(n-1)(m-1)x_{nm} - n\sum_{k=1}^{m-1} x_{nk} - m\sum_{j=1}^{n-1} x_{jm}$$

$$+\sum_{j=1}^{n-1} x_{jm} + \sum_{k=1}^{m-1} x_{nk} + \sum_{j=1}^{n-1}\sum_{k=1}^{m-1} x_{jk}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\Bigg[(n-1)(m-1)x_{nm} - (n-1)\sum_{k=1}^{m-1} x_{nk} - (m-1)\sum_{j=1}^{n-1} x_{jm} + \sum_{j=1}^{n-1}\sum_{k=1}^{m-1} x_{jk}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\sum_{j=1}^{n-1}\sum_{k=1}^{m-1}[x_{nm} - x_{nk} - x_{jm} + x_{jk}]$$

$$= \frac{1}{(n-1)(m-1)}\sum_{j=1}^{n-1}\sum_{k=1}^{m-1}\Bigg[\sum_{i=k+1}^{m} \nabla_{01}\, x_{ni} - \sum_{s=k+1}^{m} \nabla_{01}\, x_{js}\Bigg], \text{ by } D_2$$

$$= \frac{1}{(n-1)(m-1)}\sum_{j=1}^{n-1}\Bigg[\sum_{k=1}^{m-1}\sum_{i=k+1}^{m} \nabla_{01}\, x_{ni} - \sum_{k=1}^{m-1}\sum_{s=k+1}^{m} \nabla_{01}\, x_{js}\Bigg]$$

$$= \frac{1}{(n-1)(m-1)}\sum_{j=1}^{n-1}\Bigg[\sum_{i=2}^{m}(i-1)\nabla_{01}\, x_{ni} - \sum_{s=2}^{m}(s-1)\nabla_{01}\, x_{js}\Bigg]$$

$$= O(1).$$

This completes the proof of the theorem.

Corollary 8.2.3. Let $x = (x_{jk})$ be a double sequence and there exist a constant n_1 such that

(a) if $\nabla_{10}\, x_{jk} = O(\frac{1}{j})$, then $(\nabla_{10}\, C_{10}\, x)_n = O(\frac{1}{n})$

(b) if $\nabla_{01}\, x_{jk} = O(\frac{1}{k})$, then $(\nabla_{01}\, C_{01}\, x)_m = O(\frac{1}{m})$,

whenever $j, k > n_1$.

Theorem 8.2.4. For a double sequence $x = (x_{jk})$ if $st_2 - \lim C_{11}\, x = \ell$ and there exists a constant n_1 such that

(8.2.4.1) $\nabla_{11}\, x_{jk} = O(\frac{1}{jk})$;

(8.2.4.2) $\nabla_{10}\, x_{jk} = O(\frac{1}{j})$;

(8.2.4.3) $\nabla_{01}\, x_{jk} = O(\frac{1}{k})$,

whenever $j, k > n_1$. Then $\lim_{j,k} x_{jk} = \ell$.

Proof. Using conditions (8.2.4.1) – (8.2.4.3) and Lemma 8.2.2 we have $\lim C_{11}\, x = \ell$. Now by Theorem 8.1.4 we get

$$\lim_{j,k} x_{jk} = \ell.$$

Corollary 8.2.5. Let $x = (x_{jk})$ be a double sequence and there exist a constant n_1 such that

(a) if $st_2 - \lim C_{10}\, x = \ell$ and $\nabla_{10}\, x_{jk} = O(\frac{1}{j})$ then $\lim_{j,k} x_{jk} = \ell$

(b) if $st_2 - \lim C_{01}\, x = \ell$ and $\nabla_{01}\, x_{jk} = O(\frac{1}{k})$ then $\lim_{j,k} x_{jk} = \ell$,

whenever $j, k > n_1$.

Theorem 8.2.6. Let $x = (x_{jk})$ be a double sequence with $st_2 - \lim x = \ell$ and there exist constants n_1 and M such that

(8.2.6.1) $jk\nabla_{11}\, x_{j+1,k+1} \geq -M$;

(8.2.6.2) j $\nabla_{10}\, x_{j+1,k} \geq -M$;

(8.2.6.3) k $\nabla_{01}\, x_{j,k+1} \geq -M$,

whenever $j, k > n_1$. Then $\lim_{j,k} x_{jk} = \ell$.

Proof. From Theorem 8.2.1 we have

$$\lim_j \frac{\mu(j)}{\lambda(j)} = 0, \ \lim_k \frac{\mu(k)}{\lambda(k)} = 0. \tag{8.2.6.4}$$

and

$$\delta_2\{(j,k), j \leq n, k \leq m : y_{jk} = x_{jk}\} = 1$$

where $j = \lambda(j) + \mu(j)$ and $k = \lambda(k) + \mu(k)$.

Now

$$y_{\lambda(j),\lambda(k)} - x_{jk} = x_{\lambda(j),\lambda(k)} - x_{\lambda(j)+\mu(j),\lambda(k)+\mu(k)}$$

$$= -\sum_{i=\lambda(j)+1}^{\lambda(j)+\mu(j)} \nabla_{10}\, x_{i,\lambda(k)} - \sum_{s=\lambda(k)+1}^{\lambda(k)+\mu(k)} \nabla_{01}\, x_{\lambda(j)+\mu(j),s} \ \text{(by } D_1\text{)}$$

$$\leq \sum_{i=\lambda(j)+1}^{\lambda(j)+\mu(j)} \frac{b}{i-1} + \sum_{s=\lambda(k)+1}^{\lambda(k)+\mu(k)} \frac{c}{s-1}$$

$$\mid y_{\lambda(j),\lambda(k)} - x_{jk} \mid \leq \frac{b\mu(j)}{\lambda(j)} + \frac{c\mu(k)}{\lambda(k)}$$

$$\to 0 \ \text{ as } \ j,k \to \infty \ \text{ by (8.2.6.4)}.$$

Hence $\lim_{j,k} x_{jk} = \ell$.

Lemma 8.2.7. Let $x = (x_{jk})$ be a double sequence and there exist constants n_1 and M such that

(8.2.7.1) $jk\nabla_{11}\ x_{j+1,k+1} \geq -M$;

(8.2.7.2) j $\nabla_{10}\ x_{j+1,k} \geq -M$;

(8.2.7.3) k $\nabla_{01}\ x_{j,k+1} \geq -M$,

whenever $j, k > n_1$. Then $(\nabla_{11}\ C_{11}\ x)_{nm} \geq -M$.

Proof. As in Lemma 8.2.2 we have

$$(\nabla_{11}\ C_{11}\ x)_{nm} = \frac{1}{(n-1)(m-1)} \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} [x_{nm} - x_{nk} - x_{jm} - x_{jk}]$$

$$= \frac{1}{(n-1)(m-1)} \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} \left[\sum_{i=j+1}^{n} \sum_{s=k+1}^{m} \nabla_{11}\ x_{is} \right] \quad \text{(by } D_5\text{)}$$

$$= \frac{1}{(n-1)(m-1)} \sum_{i=2}^{n} \sum_{s=2}^{m} (i-1)(s-1) \nabla_{11}\ x_{is}$$

$$\geq \frac{1}{(n-1)(m-1)} \sum_{i=2}^{n} \sum_{s=2}^{m} (i-1)(s-1) \frac{-M}{(i-1)(s-1)} = -M.$$

Corollary 8.2.8. Let $x = (x_{jk})$ be a double sequence and there exist constants n_1 and M such that

(a) if $j\nabla_{10}\ x_{j+1,k} \geq -M$, then $(\nabla_{10}\ C_{10}\ x)_n \geq -M$

(b) if $k\nabla_{01}\ x_{j,k+1} \geq -M$, then $(\nabla_{01}\ C_{01}\ x)_m \geq -M$,

whenever $j, k > n_1$.

Theorem 8.2.9. Let $x = (x_{jk})$ be a double sequence with $st_2 - \lim C_{11}\ x = \ell$ and there exist constants n_1 and M such that

(8.2.9.1) $jk\nabla_{11}\ x_{j+1,k+1} \geq -M$;

(8.2.9.2) $j\ \nabla_{10}\ x_{j+1,k} \geq -M$;

(8.2.9.3) $k\ \nabla_{01}\ x_{j,k+1} \geq -M$,

whenever $j, k > n_1$. Then $\lim_{j,k} x_{jk} = \ell$.

Proof. Replacing x by $C_{11}\ x$ in Theorem 8.2.6, we get $\lim C_{11}\ x = \ell$. Now using Theorem 8.1.3 we get $\lim_{j,k} x = \ell$.

Corollary 8.2.10. Let $x = (x_{jk})$ be a double sequence and there exist constants n_1 and M such that

(a) if $st_2 - \lim C_{10}\ x = \ell$ and $j\nabla_{10}\ x_{j+1,k} \geq -M$ then $\lim_{j,k} x_{jk} = \ell$

(b) if $st_2 - \lim C_{01}\ x = \ell$ and $k\nabla_{01}\ x_{j,k+1} \geq -M$ then $\lim_{j,k} x_{jk} = \ell$,

whenever $j, k > n_1$.

8.3. Examples

In this section we give examples (i) to show that all the conditions in Theorem 8.2.1 and 8.2.4 must be satisfied, (ii) as suggested by Moricz [39, Problem 1].

Example 8.3.1. Let $x = (x_{jk})$ be a double sequence defined by

$$x_{jk} = \begin{cases} k & \text{, if } k \text{ is squqre, for all } j, \\ k^{-1} & \text{, otherwise.} \end{cases}$$

We see that

$$\mid \{(j,k) : \mid x_{jk} - 0 \mid \geq \varepsilon\} \mid \leq n\ \sqrt{m}.$$

Hence $st_2 - \lim_{j,k} x_{jk} = 0$. Also $jk\nabla_{11}\ x_{j,k} = 0 = j\nabla_{10}\ x_{j,k},\quad \forall\ j,k > 1$ but $k\nabla_{01}\ x_{j,k}$ is unbounded. Further $\lim_{j,k} x_{jk}$ does not exist. Therefore all the conditions in Theorem 8.2.1 and 8.2.4 must hold.

Note that $\lim C_{11}\ x$ does not exist but $st_2 - \lim C_{11}\ x = 0$.

Example 8.3.2. Let $x = (x_{jk})$ be a double sequence defined by

$$x_{jk} = \begin{cases} 1 & , \text{ if } j \text{ is odd, for all } k, \\ 0 & , \text{ otherwise.} \end{cases}$$

It is clear that $\lim_{j,k} x_{jk}$ and $st_2 - \lim_{j,k} x_{jk}$ do not exist but $\lim C_{11}\ x = \frac{1}{2} = st_2 - \lim C_{11}\ x$.

Now

$$jk\nabla_{11}\ x_{j,k} = 0.\quad k\nabla_{01}\ x_{j,k} = 0$$

but $j\nabla_{10}\ x_{j,k}$ is unbounded. Hence we can not drop any condition of our theorems.

This example also provides the solution of Problem 1 of Moricz, i.e. there exists a double sequence $x = (x_{jk})$ which is summable C_{11} to a finite limit, conditions

$$jk\nabla_{11}\ x_{j+1,k+1} \geq -M$$

and

$$k\nabla_{01}\ x_{j,k-1} \geq -M$$

hold but the condition

$$j\nabla_{10}\ x_{j+1,k} \geq -M$$

does not hold and x fails to converge.

BIBLIOGRAPHY

[1] S. Banach, ***Theórie des opérations linéaires***, Warszawa, 1932.

[2] H. T. Bell, *Order summability and almost convergence*, Proc. Amer. Math. Soc. 38(1973), 548-552.

[3] R. C. Buck, *Generalized asymptotic density*, Amer. J. Math. 75(1953), 335-346.

[4] B. Choudhary, *An extension of Knopp's core theorem*, J. Math. Anal. Appl. 132(1988), 226-233.

[5] J. Christopher, *The asymptotic density of some k-dimensional sets*, Amer. Math. Monthly 63(1956), 399-401.

[6] J. Connor, *The statistical and strong p-Cesàro convergence of sequences*, Analysis 8(1988), 47-63.

[7] J. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull. 32(1989), 194-198.

[8] J. Connor and J. Kline, *On statistical limit points and the consistency of statistical convergence*, J. Math. Anal. Appl. 197(1996), 392-399.

[9] R. G. Cook, ***Infinite Matrices and Sequence Spaces***, Macmillan, 1950.

[10] G. Das, *Sublinear functionals and a class of conservative matrices*, Bull. Inst. Math. Acad. Sinica. 15(1987), 89-106.

[11] K. Demirci, *A-statistical core of a sequence*, Demons. Math. (to appear).

[12] J. P. Duran, *Infinite matrices and almost convergence*, Math. Z. 128(1972), 75-83.

[13] C. Eizen and G. Laush, *Infinite matrices and almost convergence*, Math. Japon. 14(1969), 137-143.

[14] H. Fast, *Sur la convergence statistique*, Colloq. Math. 2(1951), 241-244.

[15] A. R. Freedman, *Generalized limits and sequence spaces*, Bull. London Math. Soc. 13(1981), 224-228.

[16] A. R. Freedman and J. J. Sember, *On summing sequences of 0's and 1's*, Rocky Mountain J. Math. 4(3)(1981), 419-425.

[17] A. R. Freedman and J. J. Sember, *Densities and summability*, Pacific J. Math. 95(1981), 293-305.

[18] J. A. Fridy, *On statistical convergence*, Analysis 5(1985), 301-313.

[19] J. A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. 118(1993), 1187-1192.

[20] J. A. Fridy and M. K. Khan, *Statistical extension of some classical Tauberian theorems*, Proc. Amer. Math. Soc. 128(2000), 2347-2355.

[21] J. A. Fridy and C. Orhan, *Lacunary statistical summability*, J. Math. Anal. Appl. 173(1993), 497-504.

[22] J. A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. 125(1997), 3625-3631.

[23] J. A. Fridy and C. Orhan, *Statistical core theorems*, J. Math. Anal. Appl. 208(1997), 520-527.

[24] H. Halberstem and K. F. Roth, ***Sequences***, New York, Springer-Verleg, 1983.

[25] H. J. Hamilton, *Transformations of multiple sequences*, Duke Math. J. 2(1936), 29-60.

[26] G. H. Hardy, ***Divergent Series***, Univ. Press, Oxford, 1956.

[27] J. P. King, *Almost summable sequences*, Proc. Amer. Math. Soc. 17(1966), 1219-1225.

[28] K. Knopp, *Zur Theorie der Limitierungsverfahren (Erste Mitteilung)*, Math. Z. 31(1930), 115-127.

[29] E. Kolk, *The statistical convergence in Banach spaces*, Tartu ÜI. Toimetised 928(1991), 41-52.

[30] E. Kolk, *Matrix summability of statistically convergent sequences*, Analysis 13(1993),77-83.

[31] E. Kolk, *Matrix maps into the space of statistically convergent bounded sequences*, Proc. Estonian Acad. Sci. Phys. Math. 45(1996), 187-192.

[32] E. Kolk, *Inclusion relations between the statistical convergence and strong summability*, Acta et. Comm. Univ. Tartu. Math. 2(1998), 39-54.

[33] E. Landau, *Über die Bedeutung einiger neuerer Grenzwertsätze der Herren Hardy and Axer*, Prac. Mat.-Fiz. 21(1910) , 97-177.

[34] J. Li and J. A. Fridy, *Matrix transformations of statistical cores of complex sequences*, Analysis 20(2000), 15-34.

[35] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. 80(1948), 167-190.

[36] I. J. Maddox, ***Elements of Functional Analysis***, Univ. Press, Cambridge, (1970).

[37] I. J. Maddox, *Steinhaus type theorems for summability matrices*, Proc. Amer. Math. Soc. 45(1974), 209-213.

[38] I. J. Maddox, *Some analogues of Knopp's core theorem*, Internat. J. Math. Math. Sci. 2(1979), 605-614.

[39] F. Moricz. *Tauberian theorems for Cesàro summable double sequences*, Studia Math. 110(1994), 83-96.

[40] F.Moricz and B. E. Rhoades, *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc. 104(1988). 283-294.

[41] Mursaleen, *Some analogues of Knopp core theorem*, Bull. Greek Math. Soc. (accepted).

[42] I. Niven and H. S. Zuckerman, ***An Introduction to the Theory of Numbers***, John Wiley and Sons, New York, 1980.

[43] T. Neubrum, J. Smital and T. Šalát, *On the structure of the space M(0, 1)*, Rev. Roumaine Math. Pures et App. 13(1968), 337-386.

[44] C. Orhan, *Some inequalities involving sublinear functionals*, Comm. Math. Prace Mat. 31(1991), 89-96.

[45] R. H. Patterson, *Double sequence core theorems*, Internat. J. Math. Math. Sci. 22(1999), 785-793

[46] S. Pehlivan, *Strongly almost convergent sequences defined by a modulus and uniformly statistical convergence*, Soochow J. Math. 20(1994), 205-211.

[47] A. Pringsheim, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. 53(1900), 289-321.

[48] G. M. Robinson, *Divergent double sequences and series*, Trans. Amer. Math. Soc. 28(1926), 50-73.

[49] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca 30(1980), 139-150.

[50] R. Schmidt, *Über divergent Folgen und lineare Mittelbildungen*, Math. Zeit. 22(1925), 89-152.

[51] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66(1959), 361-375.

[52] A. A. Shcherbakov, *Kernels of sequences of complex numbers and their regular transformations*, Math. Notes 22(1977), 948-953.

[53] M. Steiglitz, *Eine Verallgemeinerung des Begriffs der Fastkonvergenz*, Math. Japon. 18(1973), 53-70.

[54] M. Steiglitz and H. Tietz, *Matrixtransformationen von Folgenräumen Eine Ergebnisübersicht*, Math. Z. 154(1977), 1-16.

[55] A. Zygmund, ***Trigonometric Series***, Univ. Press, Cambridge, 1959.